

# SPREADING SPEEDS AND TRANSITION FRONTS OF LATTICE KPP EQUATIONS IN TIME HETEROGENEOUS MEDIA

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**ABSTRACT.** The current paper is devoted to the study of spreading speeds and transition fronts of lattice KPP equations in time heterogeneous media. We first prove the existence, uniqueness, and stability of spatially homogeneous entire positive solutions. Next, we establish lower and upper bounds of the (generalized) spreading speed intervals. Then, by constructing appropriate sub-solutions and super-solutions, we show the existence and continuity of transition fronts with given front position functions. Also, we prove the existence of some kind of critical front.

## 1. INTRODUCTION

The current paper deals with spatial spreading speeds and transition fronts for the following discrete Fisher-KPP equation in time heterogeneous media

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + u_i(t)f(t, u_i(t)), i \in \mathbb{Z}, \quad (1.1)$$

where  $f(t, u)$  is of monostable type. More precisely, we assume

**(H0)**  $f(t, u)$  is locally Hölder continuous in  $t \in \mathbb{R}$ , Lipschitz continuous in  $u \in \mathbb{R}$ , and continuously differentiable in  $u$  for  $u \geq 0$ .

**(H1)** For each  $u \in \mathbb{R}$ ,  $f(\cdot, u) \in L^\infty(\mathbb{R})$ ;  $f(t, u) < 0$  for  $u \geq M_0$  and some  $M_0 > 0$ ;  $f_u(t, u) < 0$  for  $u \geq 0$ ; and

$$\liminf_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t f(\tau, 0) d\tau > 0. \quad (1.2)$$

**(H2)** There are  $0 < \tilde{m}_0 < \tilde{M}_0$  such that  $f(t, 0) - \tilde{M}_0 u \leq f(t, u) \leq f(t, 0) - \tilde{m}_0 u$  for  $u \geq 0$ .

Let

$$l^\infty(\mathbb{Z}) = \{u = \{u_i\}_{i \in \mathbb{Z}} : \sup_{i \in \mathbb{Z}} |u_i| < \infty\}$$

with norm  $\|u\| = \|u\|_\infty = \sup_{i \in \mathbb{Z}} |u_i|$ . By (H0), for any given  $u^0 \in l^\infty(\mathbb{Z})$  and  $s \in \mathbb{R}$ , (1.1) has a unique (local) solution  $u(t; s, u^0) = \{u_i(t; s, u^0)\}_{i \in \mathbb{Z}}$  with  $u(s; s, u^0) = u^0$ . By (H1), if  $u_i^0 \geq 0$  for all  $i \in \mathbb{Z}$ , then  $u(t; s, u^0) = \{u_i(t; s, u^0)\}_{i \in \mathbb{Z}}$  exists for all  $t \geq s$  and  $u_i(t; s, u^0) \geq 0$  for all  $i \in \mathbb{Z}$  and  $t \geq s$  (see Proposition 2.1).

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Equation (1.1) is used to model the population dynamics of species living in patchy environments in biology and ecology (see, for example, [49, 50]). The following two equations are spatially continuous counterparts of (1.1),

$$u_t = u_{xx} + uf(t, u) \quad (1.3)$$

and

$$u_t(x, t) = \int_{\mathbb{R}} \kappa(y - x)u(y, t)dy - u(x, t) + uf(t, u), \quad (1.4)$$

where  $\kappa(\cdot)$  is a nonnegative smooth function with compact support and  $\int_{\mathbb{R}} \kappa(z)dz = 1$ . (1.3) is widely used to model the population dynamics of species when the movement or internal dispersal of the organisms occurs between adjacent locations randomly in spatially continuous media, and (1.4) is often used to model the population dynamics of species when the movement or internal dispersal of the organisms occurs between adjacent as well as non-adjacent locations in spatially continuous media. The dispersal described by  $u \mapsto u_{xx}$  in (1.3) is therefore referred to as *random dispersal* and the dispersal described by  $u(x, t) \mapsto \int_{\mathbb{R}} \kappa(y - x)u(y, t)dy - u(x, t)$  in (1.4) is referred to as *nonlocal dispersal* in literature.

One of the central dynamical issues about (1.1), (1.3), and (1.4) is to know how a solution whose initial datum is strictly positive, or is nonnegative and has compact support, or is a front-like function evolves as time increases. For example, it is important to know how a solution  $u(t; s, u^0)$  of (1.1) evolves as  $t$  increases, where  $u^0$  is strictly positive (that is,  $\inf_{i \in \mathbb{Z}} u_i^0 > 0$ ), or  $u^0$  is nonnegative (that is,  $u_i^0 \geq 0$  for all  $i \in \mathbb{Z}$ ) and has compact support (that is,  $\{i \in \mathbb{Z} : u_i^0 > 0\}$  is a bound subset of  $\mathbb{Z}$ ), or  $u^0$  is nonnegative and a front-like function (that is,

$$\sup_{i \geq I} u_i^0 \rightarrow 0, \quad \inf_{i \leq -I} u_i^0 \rightarrow u_* > 0, \quad (I \rightarrow \infty)).$$

This is strongly related to the so called spatial spreading speeds and traveling wave solutions. Pioneering works on these issues are due to Fisher [16] and Kolmogorov, Petrowsky and Piscunov [27]. They studied the existence of traveling wave solutions of (1.3) when  $f(u) = 1 - u$ , that is, solutions which can be written as  $u(x, t) = \phi(x - ct)$  with  $\phi(-\infty) = 1$ ,  $\phi(+\infty) = 0$ . Fisher in [16] found traveling wave solutions  $u(x, t) = \phi(x - ct)$  ( $\phi(-\infty) = 1$ ,  $\phi(+\infty) = 0$ ) of (1.3) with  $f(u) = 1 - u$  of all speeds  $c \geq 2$  and showed that there are no such traveling wave solutions of slower speed. Kolmogorov, Petrowsky, and Piscunov in [27] proved that for any nonnegative solution  $u(x, t)$  of (1.3) with  $f(u) = 1 - u$ , if at time  $t = 0$ ,  $u$  is 1 near  $-\infty$  and 0 near  $\infty$ , then  $\lim_{t \rightarrow \infty} u(t, ct)$  is 0 if  $c > 2$  and 1 if  $c < 2$  (that is, the population invades into the region with no initial population with speed 2).  $c_* := 2$  is therefore the minimal wave speed of (1.3) with  $f(u) = 1 - u$  and is also called the spatial spreading speed of (1.3) with  $f(u) = 1 - u$ . Thanks to the works [16] and [27], (1.1), (1.3), and (1.4) with  $f$  satisfying (H1) and (H2) are called Fisher or KPP type equations.

Since the pioneering works by Fisher ([16]) and Kolmogorov, Petrowsky, Piscunov ([27]), spatial spreading speeds and traveling wave solutions of Fisher or KPP type evolution equations in spatially and temporally homogeneous media or spatially and/or temporally periodic media have been widely studied. The reader is referred to [1, 2, 6, 7, 8, 9, 18, 22, 26, 28, 29, 30, 32, 35, 36, 37, 38, 41, 42, 52, 54], etc., for the study of Fisher or KPP type equations with random dispersal in homogeneous or periodic media, to [10, 14, 15, 39, 44, 46, 47, 48], etc., for the study of Fisher or KPP type equations with nonlocal dispersal in homogeneous

or periodic media, and to [11, 12, 13, 19, 20, 21, 24, 55, 56], etc., for the study of Fisher or KPP type equations in spatially discrete homogeneous or periodic media.

The study of spatial spreading speeds and traveling wave solutions of KPP type equations with general time and/or space dependence is more recent. Quite a few works have been carried out toward the spatial spreading speeds and traveling wave solutions of KPP type equations in non-periodic heterogeneous media. For example, in [42], [43], notions of random traveling wave solutions and generalized traveling wave solutions are introduced for random KPP equations and quite general time dependent KPP equations. In [3], [4], a notion of generalized transition waves is introduced for KPP type equations with general space and time dependence. Among others, the authors of [34] proved the existence of generalized transition waves of (1.3) with general time dependent KPP nonlinearity  $f(t, u)$ . In [9], the authors studied spreading speeds of general spatially heterogeneous including space periodic Fisher-KPP reaction diffusion equations (see also [5, 17, 18, 54], etc.). Zlatos [57] established the existence of generalized transition waves of spatially inhomogeneous Fisher-KPP reaction-diffusion equations under some specific hypotheses (see (1.2)-(1.5) in [57]). In [45], the second author of the current paper together with Zhongwei Shen proved the existence, uniqueness, and stability of generalized transition waves of (1.4) with time dependent KPP nonlinearity  $f(t, u)$  under some assumptions (see (H1), (H2) in [45]). The authors of [31] obtained the existence of generalized transition waves of spatially heterogeneous KPP equations with nonlocal dispersal under some specific assumptions (see (J1), (J2), (F1)-(F3), (G1), (G2) in [31]). For spatially discrete KPP equations, the work [40] studied spatial spreading speeds of (1.1) with time recurrent KPP nonlinearity  $f(t, u)$ . However, there is little study on spatial spreading speeds and traveling wave solutions of spatially discrete KPP type equations with general time and/or space dependence.

In this paper, we are going to investigate spatial spreading speeds and generalized transition waves for general time dependent KPP model (1.1). Throughout this paper, we assume (H0)-(H2).

We first study the existence, uniqueness, and stability of spatially homogeneous entire solutions. A solution  $u(t) = \{u_j(t)\}$  of (1.1) is called an *entire positive solution* if it is a solution of (1.1) for  $t \in \mathbb{R}$  and  $\inf_{t \in \mathbb{R}, i \in \mathbb{Z}} u_i(t) > 0$ . A solution  $u(t) = \{u_i(t)\}$  of (1.1) is called *spatially homogeneous* if  $u_i(t) = u_j(t)$  for all  $i, j \in \mathbb{Z}$ . For given function  $t \mapsto u(t) \in l^\infty(\mathbb{Z})$  and  $c \in \mathbb{R}$ , we define

$$\begin{aligned} \limsup_{|i| \geq ct, t \rightarrow \infty} u_i(t) &= \limsup_{t \rightarrow \infty} \sup_{i \in \mathbb{Z}, |i| \geq ct} u_i(t), \\ \liminf_{|i| \leq ct, t \rightarrow \infty} u_i(t) &= \liminf_{t \rightarrow \infty} \inf_{i \in \mathbb{Z}, |i| \leq ct} u_i(t), \end{aligned}$$

and

$$\limsup_{|i| \leq ct, t \rightarrow \infty} u_i(t) = \limsup_{t \rightarrow \infty} \sup_{i \in \mathbb{Z}, |i| \leq ct} u_i(t).$$

We prove

**Theorem 1.1.** (1) *There is a unique spatially homogeneous entire positive solution  $u^+(t)$  of (1.1) which is globally stable in the sense that for any  $u^0 \in l^\infty(\mathbb{Z})$  with  $\inf_{i \in \mathbb{Z}} u_i^0 > 0$ ,*

$$\|u(t+s; s, u^0) - u^+(t+s)\|_\infty \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

*uniformly in  $s \in \mathbb{R}$ .*

(2) Let  $u^0 \in l^\infty(\mathbb{Z})$  with  $u_i^0 \geq 0$  for  $i \in \mathbb{Z}$  and  $\gamma' > 0$  be given. If

$$\liminf_{|i| \leq \gamma' t, t \rightarrow \infty} u_i(t; 0, u^0) > 0,$$

then for any  $0 < \gamma < \gamma'$ ,

$$\limsup_{|i| \leq \gamma t, t \rightarrow \infty} |u_i(t; 0, u^0) - u^+(t)| = 0. \quad (1.5)$$

If

$$\liminf_{s \in \mathbb{R}, |i| \leq \gamma' t, t \rightarrow \infty} u_i(t + s; s, u^0) > 0,$$

then for any  $0 < \gamma < \gamma'$ ,

$$\limsup_{|i| \leq \gamma t, t \rightarrow \infty} |u_i(t + s; s, u^0) - u^+(t + s)| = 0 \quad (1.6)$$

uniformly in  $s \in \mathbb{R}$ .

Thanks to Theorem 1.1 (1),  $f$  satisfying (H1) and (H2) is also said to be of *monostable type*.

Next, we investigate spatial spreading speeds of (1.1). Let

$$l_0^\infty(\mathbb{Z}) = \{u = \{u_i\}_{i \in \mathbb{Z}} \in l^\infty(\mathbb{Z}) : u_i \geq 0 \text{ for all } i \in \mathbb{Z}, u_i = 0 \text{ for } |i| \gg 1, \{u_i\} \neq 0\}.$$

**Definition 1.1** (Spreading speed interval). (1) Let

$$c^* = \inf\{c : \limsup_{|i| \geq ct, t \rightarrow \infty} u_i(t; 0, u^0) = 0 \text{ for all } u^0 \in l_0^\infty(\mathbb{Z})\},$$

$$c_* = \sup\{c : \limsup_{|i| \leq ct, t \rightarrow \infty} |u_i(t; 0, u^0) - u^+(t)| = 0 \text{ for all } u^0 \in l_0^\infty(\mathbb{Z})\}.$$

We call  $[c_*, c^*]$  the spreading speed interval of (1.1).

(2) Let

$$\tilde{c}^* = \inf\{c : \limsup_{|i| \geq ct, t \rightarrow \infty} u_i(t + s; s, u^0) = 0 \text{ uniformly in } s \in \mathbb{R} \text{ for all } u^0 \in l_0^\infty(\mathbb{Z})\},$$

$$\tilde{c}_* = \sup\{c : \limsup_{|i| \leq ct, t \rightarrow \infty} |u_i(t + s; s, u^0) - u^+(t + s)| = 0 \text{ uniformly in } s \in \mathbb{R} \text{ for all } u^0 \in l_0^\infty(\mathbb{Z})\}.$$

We call  $[\tilde{c}_*, \tilde{c}^*]$  the generalized spreading speed interval of (1.1).

Observe that

$$\tilde{c}_* \leq c_* \leq c^* \leq \tilde{c}^*,$$

and that for any  $u^0 \in l_0^\infty(\mathbb{Z})$ ,

$$\limsup_{|i| \leq ct, t \rightarrow \infty} |u_i(t; 0, u^0) - u^+(t)| = 0 \quad \forall c < c_*,$$

$$\limsup_{|i| \leq ct, t \rightarrow \infty} |u_i(t + s; s, u^0) - u^+(t + s)| = 0 \quad \text{uniformly in } s \in \mathbb{R} \quad \forall c < \tilde{c}_*,$$

$$\limsup_{|i| \geq ct, t \rightarrow \infty} u_i(t; 0, u^0) = 0 \quad \forall c > c^*,$$

and

$$\limsup_{|i| \geq ct, t \rightarrow \infty} u_i(t+s; s, u^0) = 0 \quad \text{uniformly in } s \in \mathbb{R} \quad \forall c > \tilde{c}^*.$$

If  $f(t, u)$  is independent of  $t$  or periodic in  $t$ , then

$$\tilde{c}_* = c_* = c^* = \tilde{c}^*$$

(see [29, 30, 54]).

Define

$$\begin{aligned} \bar{f}_{\inf} &= \liminf_{t \geq s, t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t f(\tau, 0) d\tau, \\ \bar{f}_{\sup} &= \limsup_{t \geq s, t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t f(\tau, 0) d\tau, \\ \bar{f}_{\inf}^+ &= \liminf_{t \geq s \geq 0, t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t f(\tau, 0) d\tau, \end{aligned}$$

and

$$\bar{f}_{\sup}^+ = \limsup_{t \geq s \geq 0, t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t f(\tau, 0) d\tau.$$

We have the following theorem on the lower and upper bounds of the spreading speed intervals of (1.1).

**Theorem 1.2.** *Assume (H0)-(H2).*

$$\begin{aligned} (1) \quad c_0^- &:= \inf_{\mu > 0} \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_{\inf}^+}{\mu} \leq c_* \leq c^* \leq c_0^+ := \inf_{\mu > 0} \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_{\sup}^+}{\mu}, \\ (2) \quad \tilde{c}_0^- &:= \inf_{\mu > 0} \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_{\inf}}{\mu} \leq \tilde{c}_* \leq \tilde{c}^* \leq \tilde{c}_0^+ := \inf_{\mu > 0} \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_{\sup}}{\mu}. \end{aligned}$$

**Remark 1.1.** (1) *If  $f(\cdot, 0)$  is unique ergodic, then the limit*

$$\hat{f} = \lim_{t \geq s, t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t f(\tau, 0) d\tau$$

*exists (see [40, 43] for details). Thus*

$$c_* = c^* = \tilde{c}_* = \tilde{c}^* = \inf_{\mu > 0} \frac{e^{-\mu} + e^{\mu} - 2 + \hat{f}}{\mu}.$$

*In this case,  $c_*$  is called the spreading speed of (1.1).*

(2) *There is a unique  $\mu^* > 0$  such that*

$$\tilde{c}_0^- = \frac{e^{-\mu^*} + e^{\mu^*} - 2 + \bar{f}_{\inf}}{\mu^*}$$

*and for any  $\gamma > \tilde{c}_0^-$ , the equation  $\gamma = \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_{\inf}}{\mu}$  has exactly two positive solutions for  $\mu$  (see Lemma 5.1).*

(3) *It will be proved in Theorem 1.3 that  $\tilde{c}_* = \tilde{c}_0^-$ .*

We then study transition front solutions of (1.1).

**Definition 1.2** (Transition front). *An entire solution  $u(t) = \{u_i(t)\}_{i \in \mathbb{Z}}$  of (1.1) is called a transition front (connecting 0 and  $u^+(t)$ ) if  $u_i(t) \in (0, u^+(t))$  for all  $t \in \mathbb{R}$  and  $i \in \mathbb{Z}$ , and there exists  $J : \mathbb{R} \rightarrow \mathbb{Z}$  such that*

$$\lim_{i \rightarrow -\infty} (u_{i+J(t)}(t) - u^+(t)) = 0 \text{ and } \lim_{i \rightarrow \infty} u_{i+J(t)}(t) = 0 \text{ uniformly in } t \in \mathbb{R}.$$

The notion of a transition front is a proper generalization of a traveling wave in homogeneous media or a periodic (or pulsating) traveling wave in periodic media. The *interface location function*  $J(t)$  tells the position of the transition front  $u(t)$  as time  $t$  elapses. Notice, if  $\xi(t)$  is a bounded integer-valued function, then  $J(t) + \xi(t)$  is also an interface location function. Thus, interface location function is not unique. But, it is easy to check that if  $\tilde{J}(t)$  is another interface location function, then  $J(t) - \tilde{J}(t)$  is a bounded integer-valued function. Hence, interface location functions are unique up to addition by bounded integer-valued functions. The uniform-in- $t$  limits shows the *bounded interface width*, that is,

$$\forall 0 < \epsilon_1 \leq \epsilon_2 < 1, \quad \sup_{t \in \mathbb{R}} \text{diam}\{i \in \mathbb{Z} | \epsilon_1 \leq u_i(t) \leq \epsilon_2\} < \infty.$$

We prove

**Theorem 1.3.** (1) *For any  $\gamma > \tilde{c}_0^-$ , let  $0 < \mu < \mu^*$  and  $c(t) = \frac{e^{-\mu} + e^{\mu} - 2 + f(t, 0)}{\mu}$  be such that  $\tilde{c}_{\inf} = \gamma$ . Then there exists a continuous function  $\phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$  with  $\phi(x, t)$  being non-increasing in  $x$  and*

$$\lim_{x \rightarrow -\infty} (\phi(x, t) - u^+(t)) = 0 \text{ and } \lim_{x \rightarrow \infty} \frac{\phi(x, t)}{e^{-\mu x}} = 1 \text{ uniformly in } t \in \mathbb{R}$$

*such that  $u(t)$  is a transition front solution of (1.1), where  $u_i(t) = \phi(i - \int_0^t c(\tau) d\tau, t)$  for  $i \in \mathbb{Z}$ .*

$$(2) \tilde{c}_* = \tilde{c}_0^-.$$

**Remark 1.2.** (1) *If  $f(t, u) \equiv f(u)$  is independent of  $t$ , then so is  $\phi(x, t)$  and hence  $u_i(t) = \phi(i - ct)$  is a traveling wave solution of (1.1) in the classical sense, where  $c = \frac{e^{-\mu} + e^{\mu} - 2 + f(0)}{\mu}$  (see Remarks 5.1 and 5.2).*

(2) *If  $f(t, u)$  is periodic in  $t$  with period  $T$ , then so is  $\phi(x, t)$  (see Remarks 5.1 and 5.2). Let  $\hat{f} = \frac{\int_0^T f(\tau, 0) d\tau}{T}$  and  $\psi(x, t) = \phi(x - \frac{\int_0^t f(\tau, 0) d\tau - \hat{f}t}{\mu}, t)$ . Then  $\psi(x, t)$  is continuous in  $x \in \mathbb{R}$  and  $t \in \mathbb{R}$ , nonincreasing in  $x$ , periodic in  $t$  with period  $T$ , and*

$$u_i(t) = \phi(i - \int_0^t c(\tau) d\tau, t) = \psi(i - ct, t),$$

*where  $c = \frac{e^{-\mu} + e^{\mu} - 2 + \hat{f}}{\mu}$ . Therefore,  $u_i(t) = \psi(i - ct, t)$  is a periodic traveling wave solution with  $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$  being continuous (see Remarks 5.1 and 5.2), which is new. Observe that [29, Theorem 4.2] and [53, Theorem 6.6] imply the existence of traveling wave solution of (1.1) of the form  $u_i(t) = \Phi(i - ct, t)$ , where for each fixed  $t$ ,  $\Phi(x, t)$  is only defined for  $x \in \{i - c(nT + t) | n \in \mathbb{Z}\}$ .*

We also prove

**Theorem 1.4.** *There is a transition front solution  $u_i^*(t)$  with interface location function  $J^*(t)$  satisfying that  $u_i^*(t)$  is nonincreasing in  $i \in \mathbb{Z}$ , and*

$$\liminf_{t-s \rightarrow \infty} \frac{J^*(t) - J^*(s)}{t - s} = \tilde{c}_0^-.$$

**Remark 1.3.** *The transition front solution in Theorem 1.4 is the analogue of critical traveling front solution in literature (see [33], [42]). It is also the analogue of the traveling wave solution with minimal wave speed in the time independent case.*

The rest of the paper is organized as follows. In Section 2, we establish some basic properties of solutions of lattice equation (1.1) for the use in later sections. We study spatially homogeneous entire positive solutions of (1.1) and prove Theorem 1.1 in Section 3. In Section 4, we investigate the (generalized) spreading speeds and prove Theorem 1.2. Section 5 is devoted to the proof of the existence of transition fronts for lattice equation (1.1).

## 2. PRELIMINARY

In this section, we present some preliminary materials to be used in later sections. We first present a comparison principle for sub-solutions and super-solutions of (1.1) and prove the convergence of solutions on compact subsets. Next, we introduce the concept of the so called part metric and prove the decreasing property of the part metric between two positive solutions of (1.1) as time increases. Finally, we present a technical lemma from [34].

First of all, consider the following space continuous version of (1.1),

$$\partial_t v(x, t) = H v(x, t) + v(x, t) f(t, v(x, t)), \quad x \in \mathbb{R}, t \in \mathbb{R}, \quad (2.1)$$

where

$$H v(x, t) = v(x + 1, t) + v(x - 1, t) - 2v(x, t), \quad x \in \mathbb{R}, t \in \mathbb{R}.$$

Recall

$$l^\infty(\mathbb{Z}) = \{u : \mathbb{Z} \rightarrow \mathbb{R} : \sup_{x \in \mathbb{Z}} |u(x)| < \infty\}.$$

Let

$$l^\infty(\mathbb{R}) = \{u : \mathbb{R} \rightarrow \mathbb{R} : \sup_{x \in \mathbb{R}} |u(x)| < \infty\}$$

with norm  $\|u\| = \sup_{x \in \mathbb{R}} |u(x)|$ . Let

$$l^{\infty,+}(\mathbb{Z}) = \{u \in l^\infty(\mathbb{Z}) : \inf_{i \in \mathbb{Z}} u_i \geq 0\}, \quad l^{\infty,+}(\mathbb{R}) = \{u \in l^\infty(\mathbb{R}) : \inf_{x \in \mathbb{R}} u(x) \geq 0\}$$

and

$$l^{\infty,++}(\mathbb{Z}) = \{u \in l^\infty(\mathbb{Z}) : \inf_{i \in \mathbb{Z}} u_i > 0\}, \quad l^{\infty,++}(\mathbb{R}) = \{u \in l^\infty(\mathbb{R}) : \inf_{x \in \mathbb{R}} u(x) > 0\}.$$

For  $u, v \in l^\infty(\mathbb{R})$  (resp.  $u, v \in l^\infty(\mathbb{Z})$ ), we define

$$u \geq v \quad \text{if} \quad u - v \in l^{\infty,+}(\mathbb{R}) \quad (\text{resp. } u - v \in l^{\infty,+}(\mathbb{Z})),$$

and

$$u \gg v \quad \text{if} \quad u - v \in l^{\infty,++}(\mathbb{R}) \quad (\text{resp. } u - v \in l^{\infty,++}(\mathbb{Z})).$$

For any  $u_0 \in l^\infty(\mathbb{R})$ , let  $u(x, t; s, u_0)$  be the solution of (2.1) with  $u(x, s; s, u_0) = u_0(x)$ , and for any  $u^0 \in l^\infty(\mathbb{Z})$ , let  $u(t; s, u^0) = \{u_i(t; s, u^0)\}_{i \in \mathbb{Z}}$  be the solution of (1.1) with  $u_i(s; s, u^0) =$

$u_i^0$  for  $i \in \mathbb{Z}$ . Observe that for given  $u_0 \in l^\infty(\mathbb{R})$  and  $x_0 \in \mathbb{R}$ ,  $u(x_0 + i, t; s, u_0)$  only depends on  $\{u_0(x_0 + i) | i \in \mathbb{Z}\}$  and  $u(x_0 + i, t; s, u_0) = u_i(t; s, u^0)$ , where  $u_i^0 = u_0(x_0 + i)$  for  $i \in \mathbb{Z}$ .

A function  $v(x, t)$  on  $\mathbb{R} \times [s, T]$  which is continuous in  $t$  is called a *super-solution* or *sub-solution* of (2.1) (resp. (1.1)) if for any given  $x \in \mathbb{R}$  (resp.  $x \in \mathbb{Z}$ ),  $v(x, t)$  is absolutely continuous in  $t \in [s, T]$ , and

$$v_t(x, t) \geq H v(x, t) + v(x, t) f(t, v(x, t)) \quad \text{for a.e. } t \in [s, T]$$

or

$$v_t(x, t) \leq H v(x, t) + v(x, t) f(t, v(x, t)) \quad \text{for a.e. } t \in [s, T].$$

**Proposition 2.1** (Comparison principle). (1) If  $u_1(x, t)$  and  $u_2(x, t)$  are bounded sub-solution and super-solution of (2.1) (resp. (1.1)) on  $[s, T]$ , respectively, and  $u_1(\cdot, 0) \leq u_2(\cdot, 0)$ , then  $u_1(\cdot, t) \leq u_2(\cdot, t)$  for  $t \in [s, T]$ .

(2) Suppose that  $u_1(x, t)$ ,  $u_2(x, t)$  are bounded and satisfy that for any given  $x \in \mathbb{R}$  (resp.  $x \in \mathbb{Z}$ ),  $u_1(x, t)$  and  $u_2(x, t)$  are absolutely continuous in  $t \in [s, \infty)$ , and

$$\partial_t u_2(x, t) - (H u_2(x, t) + u_2(x, t) f(t, u_2(x, t))) > \partial_t u_1(x, t) - (H u_1(x, t) + u_1(x, t) f(t, u_1(x, t)))$$

for a.e.  $t > s$ . Moreover, suppose that  $u_2(\cdot, s) \geq u_1(\cdot, s)$ . Then  $u_2(x, t) > u_1(x, t)$  for  $x \in \mathbb{R}$  (resp.  $x \in \mathbb{Z}$ ),  $t > s$ .

(3) If  $u_0 \in l^{\infty,+}(\mathbb{R})$  (resp.  $u^0 \in l^{\infty,+}(\mathbb{Z})$ ), then  $u(x, t; s, u_0)$  (resp.  $u(t; s, u^0)$ ) exists and  $u(\cdot, t; s, u_0) \geq 0$  (resp.  $u(t; s, u^0) \geq 0$ ) for all  $t \geq s$ .

*Proof.* We prove the proposition for (2.1). It can be proved similarly for (1.1).

(1) We prove (1) by modifying the arguments of [25, Proposition 2.4].

Let  $w(x, t) = e^{ct}(u_2(x, t) - u_1(x, t))$ , where  $c$  is a constant to be determined later. Then for any given  $x \in \mathbb{R}$ , there is a measurable subset  $E$  of  $[s, T]$  with Lebesgue measure 0 such that

$$\begin{aligned} \partial_t w(x, t) &\geq H w(x, t) + (a(x, t) + c)w(x, t) \\ &= w(x + 1, t) + w(x - 1, t) + (a(x, t) - 2 + c)w(x, t) \end{aligned} \quad (2.2)$$

for  $t \in [s, T] \setminus E$ , where

$$a(x, t) = f(t, u_2(x, t)) + u_1(x, t) \int_0^1 f_u(t, s u_1(x, t) + (1 - s) u_2(x, t)) ds \quad \text{for } x \in \mathbb{R}, t \in [s, T].$$

Let  $p(x, t) = a(x, t) - 2 + c$ . By the boundedness of  $u_1$  and  $u_2$ , we can choose  $c > 0$  such that

$$\inf_{(x, t) \in \mathbb{R} \times [s, T]} p(x, t) > 0.$$

We claim that  $w(x, t) \geq 0$  for  $x \in \mathbb{R}$  and  $t \in [s, T]$ .

Let  $p_0 = \sup_{(x, t) \in \mathbb{R} \times [s, T]} p(x, t)$ . It suffices to prove the claim for  $x \in \mathbb{R}$  and  $t \in (s, T_0]$  with  $T_0 = s + \min(T - s, \frac{1}{p_0 + 2})$ . Assume that there are  $\tilde{x} \in \mathbb{R}$  and  $\tilde{t} \in (s, T_0]$  such that  $w(\tilde{x}, \tilde{t}) < 0$ . Then there is  $t^0 \in (s, T_0)$  such that

$$w_{\inf} := \inf_{(x, t) \in \mathbb{R} \times [s, t^0]} w(x, t) < 0.$$

Observe that there are  $x_n \in \mathbb{R}$  and  $t_n \in (s, t^0]$  such that

$$w(x_n, t_n) \rightarrow w_{\inf} \quad \text{as } n \rightarrow \infty.$$



By (2.2) and the fundamental theorem of calculus for Lebesgue integrals, we get

$$\begin{aligned} w(x_n, t_n) - w(x_n, s) &\geq \int_s^{t_n} [w(x_n + 1, t) + w(x_n - 1, t) + p(x_n, t)w(x_n, t)]dt \\ &\geq \int_s^{t_n} [2w_{\inf} + p(x_n, t)w_{\inf}]dt \\ &\geq (t^0 - s)(2 + p_0)w_{\inf} \quad \text{for } n \geq 1. \end{aligned}$$

Note that  $w(x_n, s) \geq 0$ , we then have

$$w(x_n, t_n) \geq (t^0 - s)(2 + p_0)w_{\inf} \quad \text{for } n \geq 1.$$

Letting  $n \rightarrow \infty$ , we obtain

$$w_{\inf} \geq (t^0 - s)(2 + p_0)w_{\inf} > w_{\inf}.$$

A contradiction. Hence the claim is true and  $u_1(x, t) \leq u_2(x, t)$  for  $x \in \mathbb{R}$  and  $t \in [s, T]$ .

(2) By the similar arguments as getting (2.2), we can find  $c, \mu > 0$  such that for any given  $x \in \mathbb{R}$ ,

$$\partial_t w(x, t) > w(x + 1, t) + w(x - 1, t) + \mu w(x, t) \quad \text{for a.e. } t > s,$$

where  $w(x, t) = e^{ct}(u_2(x, t) - u_1(x, t))$ . Then we have that for any given  $x \in \mathbb{R}$ ,

$$w(x, t) > w(x, s) + \int_s^t (w(x + 1, \tau) + w(x - 1, \tau) + \mu w(x, \tau))d\tau.$$

By the arguments in (1),  $w(x, t) \geq 0$  for all  $x \in \mathbb{R}$  and  $t \geq s$ . It then follows that  $w(x, t) > w(x, s) \geq 0$  and hence  $u_2(x, t) > u_1(x, t)$  for all  $x \in \mathbb{R}$  and  $t > s$ .

(3) By (1) and (H1), for any  $u_0 \in l^{\infty,+}(\mathbb{R})$ ,  $0 \leq u(\cdot, t; s, u_0) \leq \max\{\|u_0\|, M_0\}$  for all  $t > s$  in the existence interval of  $u(\cdot, t; s, u_0)$ . It then follows that  $u(\cdot, t; s, u_0)$  exists and  $u(\cdot, t; s, u_0) \geq 0$  for all  $t \geq s$ .  $\square$

**Proposition 2.2.** *Suppose that  $u_{0n}, u_0 \in l^{\infty,+}(\mathbb{R})$  ( $n = 1, 2, \dots$ ) with  $\{\|u_{0n}\|\}$  being bounded. If  $u_{0n}(x) \rightarrow u_0(x)$  as  $n \rightarrow \infty$  uniformly in  $x$  on bounded sets, then for each  $t > 0$ ,  $u(x, s + t; s, u_{0n}) - u(x, s + t; s, u_0) \rightarrow 0$  as  $n \rightarrow \infty$  uniformly in  $x$  on bounded sets and  $s \in \mathbb{R}$ .*

*Proof.* It can be proved by the similar arguments in [28, Proposition 3.3]. For the completeness, we provide a proof in the following.

Let  $v^n(x, t; s) = u(x, s + t; s, u_{0n}) - u(x, s + t; s, u_0)$ . Then  $v^n(t, x; s)$  satisfies

$$v_t^n(x, t; s) = H v^n(x, t; s) + a_n(t, x; s) v^n(x, t; s),$$

where

$$\begin{aligned} a_n(t, x; s) &= f(s + t, u(x, s + t; s, u_{0n})) \\ &\quad + u(x, s + t; s, u_0) \cdot \int_0^1 f_u(s + t, ru(x, s + t; s, u_{0n}) + (1 - r)u(x, s + t; s, u_0))dr. \end{aligned}$$

Observe that  $\{a_n(t, x; s)\}$  is uniformly bounded.

Take a  $\lambda > 0$ . Let

$$X(\lambda) = \{u : \mathbb{R} \rightarrow \mathbb{R} \mid u(\cdot)e^{-\lambda|\cdot|} \in l^\infty(\mathbb{R})\}$$

with norm  $\|u\|_\lambda = \|u(\cdot)e^{-\lambda|\cdot|}\|_{l^\infty(\mathbb{R})}$ . Note that  $H : X(\lambda) \rightarrow X(\lambda)$  generates an analytic semigroup, and there are  $M > 0$  and  $\omega > 0$  such that

$$\|e^{Ht}\|_{X(\lambda)} \leq Me^{\omega t} \quad \forall t \geq 0.$$

Hence

$$v^n(\cdot, t; s) = e^{Ht}v^n(\cdot, 0; s) + \int_0^t e^{H(t-\tau)}a_n(\tau, \cdot; s)v^n(\cdot, \tau; s)d\tau$$

and then

$$\|v^n(\cdot, t; s)\|_{X(\lambda)} \leq Me^{\omega t}\|v^n(\cdot, 0; s)\|_{X(\lambda)} + M \sup_{s \in \mathbb{R}, \tau \in [0, t], x \in \mathbb{R}} |a_n(\tau, x; s)| \int_0^t e^{\omega(t-\tau)}\|v^n(\cdot, \tau; s)\|_{X(\lambda)}d\tau.$$

By Gronwall's inequality,

$$\|v^n(\cdot, t; s)\|_{X(\lambda)} \leq e^{(\omega + M \sup_{s \in \mathbb{R}, \tau \in [0, t], x \in \mathbb{R}} |a_n(\tau, x; s)|)t} (M\|v^n(\cdot, 0; s)\|_{X(\lambda)}).$$

Note that  $\|v^n(\cdot, 0; s)\|_{X(\lambda)} \rightarrow 0$  uniformly in  $s \in \mathbb{R}$ . It then follows that

$$\|v^n(\cdot, t; s)\|_{X(\lambda)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly in  $s \in \mathbb{R}$  and then

$$u(x, s+t; s, u_{0n}) - u(x, s+t; s, u_0) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly in  $x$  on bounded sets and  $s \in \mathbb{R}$ .  $\square$

Next, we introduce the so called part metric and prove the decreasing property of the part metric between two positive solutions as time increases. For given  $u, v \in l^{\infty,+}(\mathbb{Z})$  (resp.  $u, v \in l^{\infty,+}(\mathbb{R})$ ), if

$$\{\alpha > 1 : \frac{1}{\alpha}v \leq u \leq \alpha v\} \neq \emptyset,$$

we define  $\rho(u, v)$  by

$$\rho(u, v) := \inf \left\{ \ln \alpha : \alpha > 1, \frac{1}{\alpha}v \leq u \leq \alpha v \right\}$$

and call  $\rho(u, v)$  the *part metric between  $u$  and  $v$* .

Observe that if  $u, v \in l^{\infty,++}(\mathbb{Z})$  (resp.  $u, v \in l^{\infty,++}(\mathbb{R})$ ), then  $\rho(u, v)$  is well defined. Observe also that if  $u, v \in l^{\infty,+}(\mathbb{Z})$  (resp.  $u, v \in l^{\infty,+}(\mathbb{R})$ ), and  $u_i > 0, v_i > 0$  for all  $i \in \mathbb{Z}$  (resp.  $u(x) > 0, v(x) > 0$  for all  $x \in \mathbb{R}$ ),  $\inf_{i \leq i_0} u_i > 0, \inf_{i \leq i_0} v_i > 0$  for any  $i_0 \in \mathbb{Z}$  (resp.  $\inf_{x \leq x_0} u(x) > 0, \inf_{x \leq x_0} v(x) > 0$  for any  $x_0 \in \mathbb{R}$ ), and  $\lim_{i \rightarrow \infty} \frac{u_i}{e^{-\mu i}} = \lim_{i \rightarrow \infty} \frac{v_i}{e^{-\mu i}} = 1$  (resp.  $\lim_{x \rightarrow \infty} \frac{u(x)}{e^{-\mu x}} = 1, \lim_{x \rightarrow \infty} \frac{v(x)}{e^{-\mu x}} = 1$ ) for some  $\mu > 0$ , then  $\rho(u, v)$  is also well defined.

**Proposition 2.3** (Part metric). (1) *For given  $u_0, v_0 \in l^{\infty,+}(\mathbb{R})$  with  $u_0 \neq v_0$ , if  $\rho(u_0, v_0)$  is well defined, then  $\rho(u(\cdot, t; s, u_0), u(\cdot, t; s, v_0))$  is also well defined for every  $t > s$  and  $\rho(u(\cdot, t; s, u_0), u(\cdot, t; s, v_0))$  decreases as  $t$  increases.*

(2) *For any  $\epsilon > 0, \sigma > 0, M > 0$ , and  $\tau > 0$  with  $\epsilon < M$  and  $\sigma \leq \ln \frac{M}{\epsilon}$ , there is  $\delta > 0$  such that for any  $u_0, v_0 \in l^{\infty,++}(\mathbb{R})$  with  $\epsilon \leq u_0(x) \leq M, \epsilon \leq v_0(x) \leq M$  for  $x \in \mathbb{R}$  and  $\rho(u_0, v_0) \geq \sigma$ , there holds*

$$\rho(u(\cdot, \tau + s; s, u_0), u(\cdot, \tau + s; s, v_0)) \leq \rho(u_0, v_0) - \delta \quad \forall s \in \mathbb{R}.$$

- (3) Suppose that  $u_1(x, t)$  and  $u_2(x, t)$  are two distinct positive entire solutions of (2.1) and that there are  $c(t)$  and  $\mu > 0$  such that

$$\lim_{x \rightarrow \infty} \frac{u_i(x + c(t), t)}{e^{-\mu x}} = 1$$

uniformly in  $t$  ( $i = 1, 2$ ) and for any  $x_0 \in \mathbb{R}$ ,

$$\inf_{x \leq x_0, t \in \mathbb{R}} u_i(x + c(t), t) > 0$$

for  $i = 1, 2$ . Then for any  $\tau > 0$  and  $T \in \mathbb{R}$ , there is  $\delta > 0$  such that

$$\rho(u_1(\cdot, s + \tau), u_2(\cdot, s + \tau)) < \rho(u_1(\cdot, s), u_2(\cdot, s)) - \delta$$

for  $s \leq T$ .

*Proof.* (1) Suppose that  $u_0, v_0 \in l^{\infty, +}(\mathbb{R})$  are such that  $u_0 \neq v_0$  and  $\rho(u_0, v_0)$  is well defined. Then there is  $\alpha > 1$  such that  $\rho(u_0, v_0) = \ln \alpha$  and

$$\frac{1}{\alpha} v_0 \leq u_0 \leq \alpha v_0.$$

By Proposition 2.1 and  $f_u(t, u) < 0$  for  $u \geq 0$  (which implies that  $f(t, \alpha u) \leq f(t, u) \leq f(t, \frac{1}{\alpha} u)$ ), we have

$$\frac{1}{\alpha} u(x, t; s, v_0) < u(x, t; s, \frac{1}{\alpha} v_0) \leq u(x, t; s, u_0) \leq u(x, t; s, \alpha v_0) < \alpha u(x, t; s, v_0)$$

for all  $x \in \mathbb{R}$ ,  $t > s$ . It then follows that

$$\rho(u(\cdot, t; s, u_0), u(\cdot, t; s, v_0)) < \rho(u_0, v_0)$$

for any  $t > s$  and then for any  $t_2 > t_1 \geq s$ ,

$$\begin{aligned} \rho(u(\cdot, t_2; s, u_0), u(\cdot, t_2; s, v_0)) &= \rho(u(\cdot, t_2; t_1, u(\cdot, t_1; s, u_0)), u(\cdot, t_2; t_1, u(\cdot, t_1; s, v_0))) \\ &< \rho(u(\cdot, t_1; s, u_0), u(\cdot, t_1; s, v_0)). \end{aligned}$$

(1) is thus proved.

(2) It can be proved by the similar arguments as in [28, Proposition 3.4]. For the self-completeness, we provide a proof in the following.

Let  $\epsilon > 0$ ,  $\sigma > 0$ ,  $M > 0$ , and  $\tau > 0$  be given and  $\epsilon < M$ ,  $\sigma < \ln \frac{M}{\epsilon}$ . First, note that by Proposition 2.1, there are  $\epsilon_1 > 0$  and  $M_1 > 0$  such that for any  $u_0 \in l^{\infty, ++}(\mathbb{R})$  with  $\epsilon \leq u_0(x) \leq M$  for  $x \in \mathbb{R}$ , there holds

$$\epsilon_1 \leq u(\cdot, t + s; s, u_0) \leq M_1 \quad \forall t \in [0, \tau], \quad s \in \mathbb{R}. \quad (2.3)$$

Let

$$\delta_1 = \epsilon_1^2 e^\sigma (1 - e^\sigma) \sup_{t \in \mathbb{R}, u \in [\epsilon_1, M_1 M / \epsilon]} f_u(t, u). \quad (2.4)$$

Then  $\delta_1 > 0$  and there is  $0 < \tau_1 \leq \tau$  such that

$$\frac{\delta_1}{2} \tau_1 < e^\sigma \epsilon_1 \quad (2.5)$$

and

$$\left| \frac{\delta_1}{2} t v f_u(t + s, w) \right| + \left| \frac{\delta_1}{2} t f(t + s, v - \frac{\delta_1}{2} t) \right| \leq \frac{\delta_1}{2} \quad \forall s \in \mathbb{R}, \quad t \in [0, \tau_1], \quad v, w \in [0, M_1 M / \epsilon]. \quad (2.6)$$

Let

$$\delta_2 = \frac{\delta_1 \tau_1}{2M_1}. \quad (2.7)$$

Then  $\delta_2 < e^\sigma$  and  $0 < \frac{\delta_2 \epsilon}{M} < 1$ . Let

$$\delta = -\ln\left(1 - \frac{\delta_2 \epsilon}{M}\right). \quad (2.8)$$

Then  $\delta > 0$ . We prove that  $\delta$  defined in (2.8) satisfies the property in the proposition.

For any  $u_0, v_0 \in l^{\infty,++}(\mathbb{R})$  with  $\epsilon \leq u_0(x) \leq M$  and  $\epsilon \leq v_0(x) \leq M$  for  $x \in \mathbb{R}$  and  $\rho(u_0, v_0) \geq \sigma$ , there is  $\alpha^* > 1$  such that  $\rho(u_0, v_0) = \ln \alpha^*$  and  $\frac{1}{\alpha^*} u_0 \leq v_0 \leq \alpha^* u_0$ . Note that  $e^\sigma \leq \alpha^* \leq \frac{M}{\epsilon}$ . By (1),  $\rho(u(\cdot, t; s, u_0), u(\cdot, t; s, v_0))$  is non-increasing in  $t > s$ . We prove that

$$\rho(u(\cdot, s + \tau; s, u_0), u(\cdot, s + \tau; s, v_0)) \leq \rho(u_0, v_0) - \delta \quad \forall s \in \mathbb{R}.$$

Let

$$v(x, t) = \alpha^* u(x, t; s, u_0).$$

Note that  $e^\sigma \leq \alpha^* \leq \frac{M}{\epsilon}$  and

$$\begin{aligned} v_t(x, t) &= H v(x, t) + v(x, t) f(t, u(x, t; s, u_0)) \\ &= H v(x, t) + v(x, t) f(t, v(x, t)) + v(x, t) f(t, u(x, t; s, u_0)) - v(x, t) f(t, v(x, t)) \\ &\geq H v(x, t) + v(x, t) f(t, v(x, t)) + \delta_1 \quad \forall s < t \leq s + \tau_1, \quad s \in \mathbb{R}. \end{aligned}$$

This together with (2.5), (2.6) implies that

$$(v(x, t) - \frac{\delta_1}{2}(t - s))_t \geq H(v(x, t) - \frac{\delta_1}{2}(t - s)) + (v(x, t) - \frac{\delta_1}{2}(t - s)) f(t, v(x, t) - \frac{\delta_1}{2}(t - s))$$

for  $s < t \leq s + \tau_1$ . Then by Proposition 2.1 again,

$$u(\cdot, t; s, \alpha^* u_0) \leq \alpha^* u(\cdot, t; s, u_0) - \frac{\delta_1}{2}(t - s) \quad \text{for } s < t \leq s + \tau_1.$$

By (2.7),

$$u(\cdot, s + \tau_1; s, v_0) \leq (\alpha^* - \delta_2) u(\cdot, s + \tau_1; s, u_0).$$

Similarly, it can be proved that

$$\frac{1}{\alpha^* - \delta_2} u(\cdot, s + \tau_1; s, u_0) \leq u(\cdot, s + \tau_1; s, v_0).$$

It then follows that

$$\rho(u(\cdot, s + \tau_1; s, u_0), u(\cdot, s + \tau_1; s, v_0)) \leq \ln(\alpha^* - \delta_2) = \ln \alpha^* + \ln(1 - \frac{\delta_2}{\alpha^*}) \leq \rho(u_0, v_0) - \delta.$$

and hence

$$\rho(u(\cdot, s + \tau; s, u_0), u(\cdot, s + \tau; s, v_0)) \leq \rho(u(\cdot, s + \tau_1; s, u_0), u(\cdot, s + \tau_1; s, v_0)) \leq \rho(u_0, v_0) - \delta.$$

(3) Without loss of generality, we assume that  $T = 0$  and fix any  $\tau > 0$ . Let  $\rho(t) = \rho(u_1(\cdot, t), u_2(\cdot, t))$ . Then there is  $\alpha(t) > 1$  such that  $\rho(t) = \ln \alpha(t)$ . We have

$$\frac{1}{\alpha(0)} u_2(x, 0) \leq u_1(x, 0) \leq \alpha(0) u_2(x, 0), \quad \forall x \in \mathbb{R}.$$

Note that

$$\lim_{x \rightarrow \infty} \frac{u_i(x + c(t), t)}{e^{-\mu x}} = 1, \quad (2.9)$$

uniformly in  $t$ . This implies that for any  $\epsilon > 0$  with  $\frac{1+\epsilon}{1-\epsilon} < \alpha(0) (\leq \alpha(t) \text{ for } t \leq 0)$ , there is  $M_\epsilon > 0$  such that

$$\frac{1-\epsilon}{1+\epsilon} u_2(x + c(t), t) \leq u_1(x + c(t), t) \leq \frac{1+\epsilon}{1-\epsilon} u_2(x + c(t), t) \quad (2.10)$$

for  $x \geq M_\epsilon$  and all  $t$ . Note also that there is  $\sigma_\epsilon > 0$  such that for  $x \leq M_\epsilon$  and all  $t$ , there holds,

$$u_i(x + c(t), t) \geq \sigma_\epsilon. \quad (2.11)$$

For any given  $s \leq 0$ , let  $\tilde{u}(x, t) = \alpha(s)u_2(x, t)$ . By (2.11), there is  $\delta_\epsilon > 0$  such that

$$\begin{aligned} \tilde{u}_t(x, t) &= H\tilde{u}(x, t) + \tilde{u}(x, t)f(t, u_2(x, t)) \\ &\geq H\tilde{u}(x, t) + \tilde{u}(x, t)f(t, \tilde{u}(x, t)) + \delta_\epsilon \end{aligned} \quad (2.12)$$

for  $x \leq M_\epsilon + c(t)$ ,  $s \leq t \leq s + \tau$ , and  $s \leq 0$ . Let  $\hat{u}(x, t) = u(x, t; s, \alpha(s)u_2(\cdot, s))$ . Note that

$$\hat{u}_t(x, t) = H\hat{u}(x, t) + \hat{u}(x, t)f(t, \hat{u}(x, t))$$

for all  $x \in \mathbb{R}$  and  $\tilde{u}(x, t) > \hat{u}(x, t)$  for  $x \in \mathbb{R}$  and  $t \geq s$ . Let  $w(x, t) = \tilde{u}(x, t) - \hat{u}(x, t)$ . Then

$$\begin{aligned} w_t(x, t) &\geq w(x + 1, t) + w(x - 1, t) - 2w(x, t) + \tilde{u}(x, t)f(t, \tilde{u}(x, t)) - \hat{u}(x, t)f(t, \hat{u}(x, t)) + \delta_\epsilon \\ &\geq p(x, t)w(x, t) + \delta_\epsilon \end{aligned}$$

for  $x \leq M_\epsilon + c(t)$ , where

$$p(x, t) = -2 + \left[ \tilde{u}(x, t)f(t, \tilde{u}(x, t)) - \hat{u}(x, t)f(t, \hat{u}(x, t)) \right] / [\tilde{u}(x, t) - \hat{u}(x, t)].$$

It then follows that

$$w(x, t) \geq \int_s^t e^{\int_r^t (-2+p(x, \tau)) d\tau} \delta_\epsilon dr$$

for  $x \leq M_\epsilon + c(t)$ . This implies that there is  $\tilde{\delta}_\epsilon > 0$  such that

$$\tilde{u}(x, s + \tau) \geq \hat{u}(x, s + \tau) + \tilde{\delta}_\epsilon$$

for  $x \leq M_\epsilon + c(s + \tau)$ . It follows that

$$u_1(x, s + \tau) \leq \hat{u}(x, s + \tau) \leq \alpha(s)u_2(x, s + \tau) - \tilde{\delta}_\epsilon \quad (2.13)$$

for  $x \leq M_\epsilon + c(s + \tau)$ .

By (2.10) and (2.13), there is  $0 < \delta < \alpha(0) (< \alpha(s))$  such that

$$u_1(x, s + \tau) \leq (\alpha(s) - \delta)u_2(x, s + \tau)$$

for  $x \in \mathbb{R}$  and  $s \leq 0$ . Similarly, we can prove that

$$u_1(x, s + \tau) \geq \frac{1}{\alpha(s) - \delta} u_2(x, s + \tau)$$

for all  $x \in \mathbb{R}$  and  $s \leq 0$ . (3) is thus proved.  $\square$

Finally, we present a technical lemma from [34]. Let

$$\bar{f}_T = \inf_{k \in \mathbb{N}} \frac{1}{T} \int_{(k-1)T}^{kT} f(\tau, 0) d\tau.$$

**Lemma 2.1.** (1) *Let  $B \in L^\infty(\mathbb{R})$ . Then*

$$\bar{B}_{\inf} = \sup_{A \in W^{1,\infty}(\mathbb{R})} \operatorname{ess\,inf}_{t \in \mathbb{R}} (A' + B)(t).$$

(2) *For given  $T > 0$ , there is  $A \in W^{1,\infty}((0, \infty))$  such that*

$$\operatorname{ess\,inf}_{t \in (0, \infty)} (A'(t) + f(t, 0)) = \bar{f}_T.$$

(3)

$$\bar{f}_{\inf}^+ = \lim_{T \rightarrow \infty} \inf_{t \geq 0} \frac{1}{T} \int_t^{t+T} f(\tau, 0) d\tau$$

and

$$\bar{f}_{\inf} = \lim_{T \rightarrow \infty} \inf_{t \in \mathbb{R}} \frac{1}{T} \int_t^{t+T} f(\tau, 0) d\tau$$

*Proof.* (1) It follows from [34, Lemma 3.2].

(2) It follows from [34, Lemma 3.2, Remark 3.3].

(3) It follows from [34, Proposition 3.1]. □

### 3. ENTIRE POSITIVE SOLUTIONS

In this section, we study entire positive solutions of (1.1) and prove Theorem 1.1.

*Proof of Theorem 1.1.* (1) First, we consider

$$\dot{u} = uf(t, u), \quad t \in \mathbb{R}. \quad (3.1)$$

For any  $u_0 \in \mathbb{R}$ , let  $u(t; s, u_0)$  be the solution of (3.1) with  $u(s; s, u_0) = u_0$ . We prove that (3.1) has an entire solution  $u^+(t)$  with  $\inf_{t \in \mathbb{R}} u^+(t) > 0$ .

Consider the linearization of (3.1) at 0,

$$\dot{v} = f(t, 0)v, \quad t \in \mathbb{R}. \quad (3.2)$$

Let  $v(t; s, v_0)$  be the solution of (3.2) with  $v(s; s, v_0) = v_0$ . Then

$$v(t; s, v_0) = e^{\int_s^t f(\tau, 0) d\tau} v_0.$$

By (H1) we can find  $\epsilon_0 > 0$  and  $T > 0$  such that

$$\frac{\int_s^{s+T} f(\tau, 0) d\tau}{T} > \epsilon_0 \quad \forall s \in \mathbb{R}.$$

Note that for the above  $\epsilon_0 > 0$ , there is  $\delta_0 > 0$  such that

$$f(t, u) \geq f(t, 0) - \epsilon_0 \quad \text{for all } t \in \mathbb{R}, |u| \leq \delta_0.$$

Let  $v_0 > 0$  be such that  $e^{\int_s^t f(\tau, 0) d\tau} v_0 \leq \delta_0$  for all  $s \in \mathbb{R}$  and  $t \in [s, s+T]$ . Then by the comparison principle for scalar ODEs,

$$u(t; s, v_0) \geq e^{\int_s^t f(\tau, 0) d\tau - \epsilon_0(t-s)} v_0 \quad \text{for } s \in \mathbb{R}, t \in [s, s+T].$$

In particular,

$$u(s+T; s, v_0) \geq e^{\int_s^{s+T} f(\tau, 0) d\tau - \epsilon_0 T} v_0 \geq v_0.$$

By induction, we have

$$u(t; s, v_0) \geq e^{\int_{s+nT}^t f(\tau, 0) d\tau - \epsilon_0(t-s-nT)} v_0 \quad \text{for } s \in \mathbb{R}, t \in [s+nT, s+(n+1)T], \quad (3.3)$$

where  $n = 0, 1, 2, \dots$ . By (H1),  $f(t, u) < 0$  for all  $t \in \mathbb{R}$  and  $u \geq M_0$ . Then

$$u(t; s, M_0) < M_0 \quad \text{for } t > s. \quad (3.4)$$

Let

$$u^n(t) = u(t; -nT, M_0), \quad t \geq -nT.$$

Then we get

$$u(t; -(n+1)T, v_0) < u^{n+1}(t) < u^n(t), \quad t \geq -nT.$$

Let

$$u^+(t) = \lim_{n \rightarrow \infty} u^n(t).$$

We have that  $u^+(t)$  is an entire solution of (3.1) and then that  $\{u_i^+(t) = u^+(t)\}_{i \in \mathbb{Z}}$  is a spatially homogeneous solution of (1.1). By (3.3),

$$\inf_{t \in \mathbb{R}} u^+(t) > 0. \quad (3.5)$$

Hence  $\{u_i^+(t) = u^+(t)\}_{i \in \mathbb{Z}}$  is a spatially homogeneous entire positive solution of (1.1). If no confusion occurs, we may still write  $\{u_i^+(t) = u^+(t)\}_{i \in \mathbb{Z}}$  as  $u^+(t)$ .

Next, we claim that for any  $u^0 \in l^{\infty, ++}(\mathbb{Z})$ ,

$$\|u(t+s; s, u^0) - u^+(t+s)\|_{\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

uniformly in  $s \in \mathbb{R}$ . Assume that there is  $u^0 \in l^{\infty, ++}(\mathbb{Z})$  such that  $\|u(t+s; s, u^0) - u^+(t+s)\|_{\infty}$  does not converge to 0 as  $t \rightarrow \infty$  uniformly in  $s \in \mathbb{R}$ . Then there are  $\tilde{\epsilon}_0 > 0$ ,  $s_n \in \mathbb{R}$ , and  $t_n \in \mathbb{R}$  with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  such that

$$\|u(t_n + s_n; s_n, u^0) - u^+(t_n + s_n)\|_{\infty} \geq \tilde{\epsilon}_0 \quad \forall n \geq 1. \quad (3.6)$$

By Proposition 2.3(1),

$$\rho(u(t + s_n; s_n, u^0), u^+(t + s_n)) < \rho(u^0, u^+(s_n)) \quad \forall t > 0.$$

This together with (3.5) implies that there are  $0 < \epsilon < M$  such that

$$\epsilon \leq u(t + s_n; s_n, u^0) \leq M, \quad \epsilon \leq u^+(t + s_n) \leq M \quad \forall t \geq s_n, n = 1, 2, \dots \quad (3.7)$$

By (3.6), (3.7), and Proposition 2.3(2), there are  $\tilde{\sigma}_0 > 0$ ,  $\tilde{\delta}_0 > 0$ , and  $\tau > 0$  such that

$$\begin{aligned} \tilde{\sigma}_0 &\leq \rho(u(t_n + s_n; s_n, u^0), u^+(t_n + s_n)) \\ &\leq \rho(u(k\tau + s_n; s_n, u^0), u^+(k\tau + s_n)) \\ &\leq \rho(u^0, u^+(s_n)) - k\tilde{\delta}_0 \quad \forall n \geq 1, 1 \leq k \leq [t_n/\tau]. \end{aligned}$$

This is a contradiction. Hence the claim holds.

By the claim in the above, (1.1) has only one spatially homogeneous entire positive solution. (1) is thus proved.

(2) We prove (1.6). (1.5) can be proved similarly.

Let

$$\delta_0 = \liminf_{s \in \mathbb{R}, |i| \leq \gamma' t, t \rightarrow \infty} u_i(s+t; s, u^0).$$

Then there is  $T > 0$  such that

$$u_i(s+t; s, u^0) \geq \frac{\delta_0}{2} \quad \forall s \in \mathbb{R}, |i| \leq \gamma' t, t \geq T.$$

Assume that there is  $0 < \gamma_0 < \gamma'$  such that (1.6) does not hold. Then there are  $\epsilon_0 > 0$ ,  $s_n \in \mathbb{R}$ ,  $i_n \in \mathbb{Z}$ ,  $t_n > 0$  such that  $|i_n| \leq \gamma_0 t_n$ ,  $t_n \rightarrow \infty$ , and

$$|u_{i_n}(s_n + t_n; s_n, u^0) - u^+(s_n + t_n)| \geq \epsilon_0. \quad (3.8)$$

Let  $\tilde{u}^0 = \{\tilde{u}_i^0\}$  and  $\hat{u}^0 = \{\hat{u}_i^0\}$ , where  $\tilde{u}_i^0 = \frac{\delta_0}{2}$  and  $\hat{u}_i^0 = \|u^0\|$  for all  $i \in \mathbb{Z}$ . By (1), there is  $\tilde{T} \geq T$  such that

$$|u_i(s+t; s, \tilde{u}^0) - u^+(s+t)| < \frac{\epsilon_0}{2} \quad \forall i \in \mathbb{Z}, s \in \mathbb{R}, t \geq \tilde{T} \quad (3.9)$$

and

$$u_i(s+t; s, u^0) \leq u_i(s+t; s, \hat{u}^0) \leq u^+(s+t) + \epsilon_0 \quad \forall i \in \mathbb{Z}, s \in \mathbb{R}, t \geq \tilde{T}. \quad (3.10)$$

Let  $\tilde{u}^n = \{\tilde{u}_i^n\}$  be given by

$$\tilde{u}_i^n = \begin{cases} \frac{\delta_0}{2} & \forall |i| \leq (\gamma' - \gamma_0)(t_n - \tilde{T}) \\ 0 & \text{for otherwise.} \end{cases}$$

Then

$$\lim_{n \rightarrow \infty} \tilde{u}_i^n = \tilde{u}_i^0 \quad \text{locally uniformly.}$$

By Proposition 2.2,

$$\lim_{n \rightarrow \infty} (u_i(s_n + t_n; s_n + t_n - \tilde{T}, \tilde{u}^n) - u_i(s_n + t_n; s_n + t_n - \tilde{T}, \tilde{u}^0)) = 0 \quad (3.11)$$

locally uniformly in  $i \in \mathbb{Z}$ .

Observe that

$$\begin{aligned} u_{i_n}(s_n + t_n; s_n, u^0) &= u_{i_n}(s_n + t_n; s_n + t_n - \tilde{T}, u(s_n + t_n - \tilde{T}; s_n, u^0)) \\ &= u_0(s_n + t_n; s_n + t_n - \tilde{T}, u_{\cdot + i_n}(s_n + t_n - \tilde{T}; s_n, u^0)) \\ &\geq u_0(s_n + t_n; s_n + t_n - \tilde{T}, \tilde{u}^n) \quad \text{for } n \gg 1. \end{aligned}$$

This together with (3.9), (3.10), and (3.11) implies that

$$u^+(s_n + t_n) - \epsilon_0 < u_{i_n}(s_n + t_n; s_n, u^0) < u^+(s_n + t_n) + \epsilon_0 \quad (3.12)$$

for  $n \gg 1$ , which contradicts to (3.8). Hence (1.6) holds.  $\square$



## 4. SPREADING SPEEDS

In this section, we investigate spreading speeds of (1.1) and prove Theorem 1.2. First we present two lemmas.

For given  $T > 0$ , recall that

$$\bar{f}_T = \inf_{k \in \mathbb{N}} \frac{1}{T} \int_{(k-1)T}^{kT} f(\tau, 0) d\tau.$$

By Lemma 2.1(3),

$$\bar{f}_{\inf}^+ = \lim_{T \rightarrow \infty} \inf_{t \geq 0} \frac{1}{T} \int_t^{t+T} f(\tau, 0) d\tau.$$

So we have

**Lemma 4.1.** *For given  $\gamma' < c_0^-$ , there is  $T > 0$  such that*

$$\gamma' < \inf_{\mu > 0} \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_T}{\mu}.$$

**Lemma 4.2.** *For any given  $M > 0$ , consider*

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + u_i(t)(\bar{f}_T - Mu_i(t)). \quad (4.1)$$

*Let  $[c_{*,T}, c^{*,T}]$  be the spreading speed interval of (4.1). Then*

$$c_{*,T} = c^{*,T} = \inf_{\mu > 0} \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_T}{\mu}.$$

*Proof.* See [40, Theorem 2.3]. □

We now prove Theorem 1.2.

*Proof of Theorem 1.2.* (1) First we prove that for any given  $\gamma' < c_0^-$  and  $u^0 \in l_0^\infty(\mathbb{Z})$ ,

$$\liminf_{|i| \leq \gamma' t, t \rightarrow \infty} u_i(t; 0, u^0) > 0. \quad (4.2)$$

For the given  $\gamma' < c_0^-$ , let  $T > 0$  be as in Lemma 4.1 and  $A(t)$  be as in Lemma 2.1(2). Put  $v_i(t) = u_i(t; 0, u^0)e^{A(t)}$ . Then  $v_i(t)$  is absolutely continuous in and differentiable in  $t \in [0, \infty)$  and satisfies

$$\begin{aligned} \dot{v}_i(t) &= \dot{u}_i(t; 0, u^0)e^{A(t)} + A'(t)u_i(t; 0, u^0)e^{A(t)} \\ &= v_{i+1}(t) + v_{i-1}(t) - 2v_i(t) + v_i(t)(f(t, u_i(t; 0, u^0)) + A'(t)) \\ &\geq v_{i+1}(t) + v_{i-1}(t) - 2v_i(t) + v_i(t)(f(t, 0) - \tilde{M}_0 u_i(t; 0, u^0) + A'(t)) \\ &\geq v_{i+1}(t) + v_{i-1}(t) - 2v_i(t) + v_i(t)(\bar{f}_T - \tilde{M}_0 u_i(t; 0, u^0)) \\ &= v_{i+1}(t) + v_{i-1}(t) - 2v_i(t) + v_i(t)(\bar{f}_T - \tilde{M}_0 e^{-A(t)} v_i(t)) \\ &\geq v_{i+1}(t) + v_{i-1}(t) - 2v_i(t) + v_i(t)(\bar{f}_T - \tilde{M} v_i(t)) \end{aligned} \quad (4.3)$$

for a.e.  $t > 0$ , where  $\tilde{M} = \tilde{M}_0 \sup_{t > 0} e^{-A(t)}$ . By Lemmas 4.1 and 4.2,

$$\liminf_{|i| \leq \gamma' t, t \rightarrow \infty} v_i(t) > 0.$$

This implies that (4.2) holds.

For any  $\gamma < c_0^-$ , let  $\gamma' \in (\gamma, c_0^-)$ . By (4.2) and Theorem 1.1(2),

$$\limsup_{|i| \leq \gamma t, t \rightarrow \infty} |u_i(t; 0, u^0) - u^+(t)| = 0.$$

Thus  $c_0^- \leq c_*$ .

Next we prove that for any  $\gamma > c_0^+$  and  $u^0 \in l_0^\infty(\mathbb{Z})$ ,

$$\limsup_{|i| \geq \gamma t, t \rightarrow \infty} u_i(t; 0, u^0) = 0. \quad (4.4)$$

For the given  $\gamma > c_0^+$ , there is  $\tilde{T} > 0$  such that

$$\gamma > \inf_{\mu > 0} \frac{e^{-\mu} + e^\mu - 2 + \tilde{f}_{\tilde{T}}}{\mu} \quad (4.5)$$

with

$$\tilde{f}_{\tilde{T}} = \sup_{k \in \mathbb{N}} \frac{1}{\tilde{T}} \int_{(k-1)\tilde{T}}^{k\tilde{T}} f(\tau, 0) d\tau.$$

Then by Lemma 2.1(2) with  $f(t, 0)$  and  $T$  replaced by  $-f(t, 0)$  and  $\tilde{T} > 0$ , respectively, there is  $\tilde{A}(t) \in W^{1,\infty}((0, \infty))$  such that

$$-\tilde{f}_{\tilde{T}} = \inf_{k \in \mathbb{N}} \frac{1}{\tilde{T}} \int_{(k-1)\tilde{T}}^{k\tilde{T}} (-f(\tau, 0)) d\tau = \operatorname{ess\,inf}_{t \in (0, \infty)} (-\tilde{A}'(t) - f(t, 0)). \quad (4.6)$$

Put  $\tilde{v}_i(t) = u_i(t; 0, u^0)e^{\tilde{A}(t)}$ . By (H2),  $f(t, u) \leq f(t, 0) - \tilde{m}_0 u$ . Then  $\tilde{v}_i(t)$  is absolutely continuous in  $t \in [0, \infty)$  and satisfies

$$\begin{aligned} \dot{\tilde{v}}_i(t) &= \dot{u}_i(t; 0, u^0)e^{\tilde{A}(t)} + \tilde{A}'(t)u_i(t; 0, u^0)e^{\tilde{A}(t)} \\ &= \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + \tilde{v}_i(t)(f(t, u_i(t; 0, u^0)) + \tilde{A}'(t)) \\ &\leq \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + \tilde{v}_i(t)(f(t, 0) - \tilde{m}_0 u_i(t; 0, u^0) + \tilde{A}'(t)) \\ &\leq \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + \tilde{v}_i(t)(\tilde{f}_{\tilde{T}} - \tilde{m}_0 u_i(t; 0, u^0)) \\ &= \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + \tilde{v}_i(t)(\tilde{f}_{\tilde{T}} - \tilde{m}_0 e^{-\tilde{A}(t)} \tilde{v}_i(t)) \\ &\leq \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + \tilde{v}_i(t)(\tilde{f}_{\tilde{T}} - \tilde{m} \tilde{v}_i(t)) \end{aligned} \quad (4.7)$$

for a.e.  $t > 0$ , where  $\tilde{m} = \tilde{m}_0 \inf_{t > 0} e^{-\tilde{A}(t)}$ . By Lemma 4.2 and (4.5),

$$\limsup_{|i| \geq \gamma t, t \rightarrow \infty} \tilde{v}_i(t) = 0.$$

This implies that (4.4) holds. Thus  $c_0^+ \geq c^*$ .

(2) Note that from the proof of [34, Lemma 3.2], we can also get that for given  $T > 0$ , there is  $\hat{A} \in W^{1,\infty}(\mathbb{R})$  such that

$$\operatorname{ess\,inf}_{t \in \mathbb{R}} (\hat{A}'(t) + f(t, 0)) = \inf_{k \in \mathbb{Z}} \frac{1}{T} \int_{(k-1)T}^{kT} f(\tau, 0) d\tau.$$

Then the results can be proved by the similar arguments as in (1).  $\square$

## 5. TRANSITION FRONTS

In this section, we study transition fronts and prove Theorems 1.3 and 1.4. We first prove some important lemmas.

For given  $\mu > 0$ , let

$$c(t; \mu) = \frac{e^{-\mu} + e^{\mu} - 2 + f(t, 0)}{\mu}.$$

Recall that

$$\tilde{c}_0^- = \inf_{\mu > 0} \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_{\inf}}{\mu}.$$

**Lemma 5.1.** *There is a unique  $\mu^* > 0$  such that*

$$\tilde{c}_0^- = \frac{e^{-\mu^*} + e^{\mu^*} - 2 + \bar{f}_{\inf}}{\mu^*}$$

and for any  $\gamma > \tilde{c}_0^-$ , the equation  $\gamma = \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_{\inf}}{\mu}$  has exactly two positive solutions for  $\mu$ .

*Proof.* Let  $\chi_1(\mu) = \frac{e^{-\mu} + e^{\mu} - 2 + \bar{f}_{\inf}}{\mu}$  and  $\chi_2(\mu) = \frac{\partial}{\partial \mu}(\mu \chi_1(\mu))$ . Then

$$\frac{\partial \chi_2}{\partial \mu}(\mu) = e^{\mu} - e^{-\mu} > 0,$$

$$\frac{\partial \chi_1}{\partial \mu}(\mu) = \frac{1}{\mu}[\chi_2(\mu) - \chi_1(\mu)],$$

$$\frac{\partial}{\partial \mu}(\mu^2 \frac{\partial \chi_1}{\partial \mu}(\mu)) = \mu \frac{\partial \chi_2}{\partial \mu}(\mu) > 0 \text{ for } \mu > 0.$$

Hence there is at most one  $\mu > 0$  such that

$$\frac{\partial \chi_1}{\partial \mu}(\mu) = 0.$$

The lemma then follows from

$$\lim_{\mu \rightarrow +\infty} \chi_1(\mu) = +\infty,$$

and

$$\lim_{\mu \rightarrow 0^+} \chi_1(\mu) = +\infty \text{ (by (1.2)).}$$

□

**Lemma 5.2.** *For any  $\gamma > \tilde{c}_0^-$ , let  $0 < \mu < \mu^*$  be such that  $\chi_1(\mu) = \gamma$  and  $c(t) = c(t; \mu)$ . Then there are  $\bar{\phi}(x, t)$  and  $\underline{\phi}(x, t)$  satisfying the following properties.*

(1)  $\bar{\phi}(x, t)$  and  $\underline{\phi}(x, t)$  are continuous functions in  $t \in \mathbb{R}$  and  $x \in \mathbb{R}$ ,

$$0 < \underline{\phi}(x, t) < \bar{\phi}(x, t) \leq u^+(t), \quad \bar{\phi} \text{ is nonincreasing in } x \in \mathbb{R},$$

(2) For any  $M \in \mathbb{R}$ ,

$$\inf_{x \leq M, t \in \mathbb{R}} \underline{\phi}(x, t) > 0, \quad \inf_{x \leq M, t \in \mathbb{R}} \bar{\phi}(x, t) > 0.$$

(3) *The limits*

$$\lim_{x \rightarrow \infty} \frac{\bar{\phi}(x, t)}{e^{-\mu x}} = \lim_{x \rightarrow \infty} \frac{\phi(x, t)}{e^{-\mu x}} = 1$$

*exist and are uniform in  $t \in \mathbb{R}$ .*

(4) *Let*

$$\bar{v}(\cdot, t) = \bar{\phi}(\cdot - \int_0^t c(\tau) d\tau, t) \quad \text{and} \quad \underline{v}(\cdot, t) = \underline{\phi}(\cdot - \int_0^t c(\tau) d\tau, t).$$

*Then*

$$u(x, t; s, \bar{v}(\cdot, s)) \leq \bar{v}(x, t), \quad u(x, t; s, \underline{v}(\cdot, s)) \geq \underline{v}(x, t)$$

*for all  $x \in \mathbb{R}$  and  $t \geq s$ .*

*Proof.* First of all, we may assume that  $f(t, u) = f(t, 0)$  for  $u < 0$ . For otherwise, we may replace  $f(t, u)$  by  $\tilde{f}(t, u)$ , where  $\tilde{f}(t, u) = f(t, u)$  for  $u \geq 0$  and  $\tilde{f}(t, u) = f(t, 0)$  for  $u < 0$ .

We first construct  $\bar{\phi}(x, t)$  satisfying  $\bar{\phi}(x, t) \leq u^+(t)$  and (2)-(4). Let  $\varphi(x) = e^{-\mu x}$ . Then  $\varphi(x)$  is a solution of the equation

$$\begin{aligned} 0 &= \partial_t \varphi - H\varphi - c(t)\partial_x \varphi - f(t, 0)\varphi \\ &= \varphi[c(t)\mu - (e^\mu + e^{-\mu} - 2 + f(t, 0))], \quad \text{for } x \in \mathbb{R}, t \in \mathbb{R}. \end{aligned}$$

Let  $\hat{v}(x, t) = \varphi(x - \int_0^t c(\tau) d\tau) = e^{-\mu(x - \int_0^t c(\tau) d\tau)}$ . Then  $\hat{v}(x, t)$  satisfies

$$\partial_t \hat{v}(x, t) = H\hat{v}(x, t) + f(t, 0)\hat{v}(x, t) \geq H\hat{v} + \hat{v}f(t, \hat{v}), \quad x \in \mathbb{R}, t \in \mathbb{R}.$$

Thus it is a super-solution of (2.1). Moreover, for any constant  $C$ ,  $\hat{u}(x, t) := e^{Ct}\hat{v}(x, t)$  satisfies

$$\partial_t \hat{u}(x, t) = (\partial_t \hat{v}(x, t) + C\hat{v}(x, t))e^{Ct} \geq H\hat{u}(x, t) + C\hat{u}(x, t) + \hat{u}(x, t)f(t, \hat{v}(x, t)),$$

hence

$$\hat{u}(x, t) \geq \hat{u}(x, s) + \int_s^t \left( H\hat{u}(x, \tau) + C\hat{u}(x, \tau) + \hat{u}(x, \tau)f(\tau, \hat{v}(x, \tau)) \right) d\tau,$$

and  $\tilde{u}(x, t) = e^{Ct}u^+(t)$  satisfies

$$\partial_t \tilde{u}(x, t) = (\partial_t u^+(t) + Cu^+(t))e^{Ct} = H\tilde{u}(x, t) + C\tilde{u}(x, t) + \tilde{u}(x, t)f(t, u^+(t)),$$

hence

$$\tilde{u}(x, t) = \tilde{u}(x, s) + \int_s^t \left( H\tilde{u}(x, \tau) + C\tilde{u}(x, \tau) + \tilde{u}(x, \tau)f(\tau, u^+(\tau)) \right) d\tau.$$

Let

$$\bar{\phi}(x, t) = \min\{\varphi(x), u^+(t)\}$$

and

$$\bar{v}(x, t) = \bar{\phi}(x - \int_0^t c(\tau) d\tau, t).$$

It is clear that

$$\bar{\phi}(x, t) \leq u^+(t) \quad \forall x \in \mathbb{R}, t \in \mathbb{R}$$

and that  $\bar{\phi}(x, t)$  satisfies (2) and (3). We prove that  $\bar{\phi}(x, t)$  also satisfies (4).

Recall that  $\bar{v}(x, t) = \bar{\phi}(x - \int_0^t c(\tau) d\tau, t)$ . Note that for any constant  $C$ ,  $u(x, t) = e^{Ct} \bar{v}(x, t)$  satisfies

$$u(x, t) \geq u(x, s) + \int_s^t \left( Hu(x, \tau) + Cu(x, \tau) + u(x, \tau) f(\tau, \bar{v}(x, \tau)) \right) d\tau.$$

Let  $w(x, t) = e^{Ct} (\bar{v}(x, t) - u(x, t; s, \bar{v}(\cdot, s)))$ . Then

$$w(x, t) \geq w(x, s) + \int_s^t \left( Hw(x, \tau) + Cw(x, \tau) + a(x, \tau)w(x, \tau) \right) d\tau,$$

where

$$a(x, \tau) = f(\tau, u(x, \tau; s, \bar{v}(\cdot, s))) + \bar{v}(x, \tau) \int_0^1 f_u(\tau, r\bar{v}(x, \tau) + (1-r)u(x, \tau; s, \bar{v}(\cdot, s))) dr.$$

Choose  $C > 0$  such that  $C - 2 + a(x, t) > 0$  for all  $t \in \mathbb{R}$  and  $x \in \mathbb{R}$ . By the arguments of Proposition 2.1, we have

$$w(x, t) \geq w(x, s) = 0,$$

and hence

$$u(x, t; s, \bar{v}(\cdot, s)) \leq \bar{v}(x, t) \quad \forall x \in \mathbb{R}, t \geq s.$$

Hence  $\bar{\phi}(x, t)$  also satisfies (4).

Next, we construct  $\underline{\phi}(x, t)$  satisfying (1)-(4). Let  $\tilde{M}_0$  be as in (H2). Let  $B(t) = -(e^{-\tilde{\mu}} + e^{\tilde{\mu}} - 2) + c(t)\tilde{\mu} - f(t, 0)$ . Note that

$$\begin{aligned} \bar{B}_{\inf} &= -(e^{-\tilde{\mu}} + e^{\tilde{\mu}} - 2) + \gamma\tilde{\mu} - \bar{f}_{\inf} \\ &= \tilde{\mu} \left( \gamma - \frac{e^{-\tilde{\mu}} + e^{\tilde{\mu}} - 2 + \bar{f}_{\inf}}{\tilde{\mu}} \right), \end{aligned}$$

thus we can choose  $\tilde{\mu} \in (\mu, 2\mu)$  such that  $\bar{B}_{\inf} > 0$ . Due to Lemma 2.1, we can then find  $A \in W^{1,\infty}(\mathbb{R})$  such that  $\text{ess inf}_{t \in \mathbb{R}} (A' + B) > 0$ .

Let  $\psi(x, t) = e^{-\mu x} - e^{A(t) - \tilde{\mu} x}$ . Then for each  $x$ ,  $\psi(x, t)$  is absolutely continuous in  $t$ . We claim that  $\psi(x, t)$  satisfies that for each  $x \in \mathbb{R}$ ,

$$\partial_t \psi - H\psi - c(t)\partial_x \psi \leq \psi f(t, \psi) \quad (5.1)$$

for a.e.  $t \in \mathbb{R}$ . Note that for each  $x$ ,

$$\begin{aligned} &\partial_t \psi - H\psi - c(t)\partial_x \psi - f(t, 0)\psi \\ &= [-A'(t) + e^{-\tilde{\mu}} + e^{\tilde{\mu}} - 2 - c(t)\tilde{\mu} + f(t, 0)]e^{A(t) - \tilde{\mu} x} \quad \text{for a.e. } t \in \mathbb{R}. \end{aligned}$$

Note also that

$$f(t, 0) - [A'(t) - (e^{-\tilde{\mu}} + e^{\tilde{\mu}} - 2) + c(t)\tilde{\mu}] < 0 \quad \forall t \in \mathbb{R}.$$

For given  $x \in \mathbb{R}$  and  $t \in \mathbb{R}$  such that  $\partial_t \psi(x, t)$  exists, if  $\psi(x, t) \leq 0$ , then

$$\begin{aligned} &\partial_t \psi - H\psi - c(t)\partial_x \psi \\ &= f(t, 0)\psi(x, t) + [-A'(t) + e^{-\tilde{\mu}} + e^{\tilde{\mu}} - 2 - c(t)\tilde{\mu} + f(t, 0)]e^{A(t) - \tilde{\mu} x} \\ &\leq f(t, 0)\psi(x, t) \\ &= \psi(x, t)f(t, \psi(x, t)). \end{aligned}$$

On the other hand, if  $\psi(x, t) > 0$ , then  $x \geq (\tilde{\mu} - \mu)^{-1}A(t)$  and we have

$$\tilde{M}_0 \psi^2 e^{\tilde{\mu}x - A(t)} \leq \tilde{M}_0 e^{(\tilde{\mu} - 2\mu)x - A(t)} \leq \tilde{M}_0 e^{-A(t)} \quad \text{for } x \geq \max(0, (\tilde{\mu} - \mu)^{-1}A(t)), \quad t \in \mathbb{R}.$$

By adding a large constant  $\alpha$  to  $A(t)$ , we have  $(\tilde{\mu} - \mu)^{-1}A(t) > 0$  and

$$A'(t) + B(t) \geq \tilde{M}_0 \psi^2 e^{\tilde{\mu}x - A(t)} \quad \text{for a.e. } t \in \mathbb{R}. \quad (5.2)$$

This implies

$$\begin{aligned} & \partial_t \psi - H\psi - c(t)\partial_x \psi \\ &= f(t, 0)\psi(x, t) + [-A'(t) + e^{-\tilde{\mu}} + e^{\tilde{\mu}} - 2 - c(t)\tilde{\mu} + f(t, 0)]e^{A(t) - \tilde{\mu}x} \\ &\leq f(t, 0)\psi(x, t) - \tilde{M}_0 \psi^2(x, t) \\ &\leq \psi(x, t)f(t, \psi(x, t)). \end{aligned}$$

Therefore, the claim holds.

Let  $u_K^+(t)$  be the unique entire positive solution of

$$\dot{u} = u(f(t, 0) - Ku).$$

For  $K \gg 1$ ,  $\sup_{t \in \mathbb{R}} u_K^+(t) \ll \sup_{x \in \mathbb{R}, t \in \mathbb{R}} \psi(x, t)$ , and  $u_K^+(t)$  is a sub-solution of (2.1). Note that for each  $t$ , there are  $X_1(t) < X_2(t)$  such that  $u_K^+(t) = \psi(x - \int_0^t c(\tau)d\tau, t)$  for  $x = X_i(t)$  ( $i = 1, 2$ ),  $u_K^+(t) > \psi(x - \int_0^t c(\tau)d\tau, t)$  for  $x < X_1(t)$  or  $x > X_2(t)$ , and  $u_K^+(t) < \psi(x - \int_0^t c(\tau)d\tau, t)$  for  $X_1(t) < x < X_2(t)$ . When  $K \gg 1$ ,  $X_2(t) - X_1(t) > 1$ .

Let

$$\underline{\phi}(x, t) = \begin{cases} \psi(x, t), & x \geq X_1(t) - \int_0^t c(\tau)d\tau \\ u_K^+(t), & x < X_1(t) - \int_0^t c(\tau)d\tau \end{cases}$$

and  $\underline{v}(x, t) = \underline{\phi}(x - \int_0^t c(\tau)d\tau, t)$ . By the similar arguments as in the construction of  $\bar{\phi}$ ,  $u(x, t) = e^{Ct} \underline{v}(\cdot, t)$  satisfies

$$u(x, t) \leq u(x, s) + \int_s^t \left( Hu(x, \tau) + Cu(x, \tau) + u(x, \tau)f(\tau, \underline{v}(x, \tau)) \right) d\tau$$

and

$$u(x, t; s, \underline{v}(\cdot, s)) \geq \underline{v}(x, t)$$

for all  $x \in \mathbb{R}$  and  $t \geq s$ . It is clear that  $\underline{\phi}(x, t)$  satisfies (1)-(3). The lemma is thus proved.  $\square$

**Remark 5.1.** (1) If  $f(t, u) \equiv f(u)$ , then  $\bar{\phi}(x, t) \equiv \bar{\phi}(x)$ .

(2) If  $f(t, u) = f(t + T, u)$ , then  $\bar{\phi}(x, t + T) = \bar{\phi}(x, t)$ .

**Lemma 5.3.** Suppose that  $u(t) = \{u_j(t)\} \in l^\infty(\mathbb{Z})$  and  $v(t) = \{v_j(t)\} \in l^\infty(\mathbb{Z})$  are nonnegative solutions of (1.1) on  $[t_0, \infty)$  and  $u_j(t_0) \neq v_j(t_0)$ . If there is  $j_0$  such that  $u_j(t_0) \geq v_j(t_0)$  for  $j \leq j_0$  and  $u_j(t_0) \leq v_j(t_0)$  for  $j > j_0$ , then for any  $t > t_0$ , there is  $j_t \in \mathbb{Z} \cup \{-\infty, \infty\}$  such that  $u_j(t) \geq v_j(t)$  for  $j \leq j_t$  and  $u_j(t) \leq v_j(t)$  for  $j > j_t$ . Moreover, if  $u_{i_t}(t) = v_{i_t}(t)$  for some  $i_t \in \mathbb{Z}$  and  $t \in (t_0, \infty)$ , then  $u_j(t) \geq v_j(t)$  for  $j \leq i_t$  and  $u_j(t) \leq v_j(t)$  for  $j > i_t$ .

*Proof.* If  $u_j(t_0) = v_j(t_0)$  for all  $j \leq j_0$ , then  $u_j(t_0) \leq v_j(t_0)$  for all  $j \in \mathbb{Z}$ . By Proposition 2.1,  $u_j(t) < v_j(t)$  for all  $t > t_0$  and  $j \in \mathbb{Z}$ . The lemma then follows.

Similarly, if  $u_j(t_0) = v_j(t_0)$  for all  $j > j_0$ , then  $u_j(t) > v_j(t)$  for all  $t > t_0$  and  $j \in \mathbb{Z}$ . The lemma also follows.

Assume that there are  $j_1 \leq j_0$  and  $j_2 > j_0$  such that  $u_{j_1}(t_0) > v_{j_1}(t_0)$  and  $u_{j_2}(t_0) < v_{j_2}(t_0)$ . Without loss of generality, we may assume that  $j_1 = j_0$ . Then there is  $\epsilon > 0$  such that  $u_{j_0}(t) > v_{j_0}(t)$  for  $t_0 \leq t \leq t_0 + \epsilon$ . It follows from the arguments of Proposition 2.1(1) that

$$u_j(t) > v_j(t) \quad \text{for } t \in (t_0, t_0 + \epsilon], \quad j \leq j_0,$$

and

$$u_j(t) < v_j(t) \quad \text{for } t \in (t_0, t_0 + \epsilon], \quad j > j_0.$$

By [23, Lemma 4], for any  $t > t_0$ , there is  $j_t \in \mathbb{Z} \cup \{-\infty, \infty\}$  such that  $u_j(t) \geq v_j(t)$  for  $j \leq j_t$  and  $u_j(t) \leq v_j(t)$  for  $j > j_t$ .

Moreover, suppose that  $u_{i_t}(t) = v_{i_t}(t)$  for some  $t \in (t_0, \infty)$  and  $i_t \in \mathbb{Z}$ . We claim that  $u_j(t) \geq v_j(t)$  for  $j \leq i_t$ . For otherwise, assume that there is  $j^* < i_t$  such that  $u_{j^*}(t) < v_{j^*}(t)$ . Then  $j_t \leq j^*$  and  $u_j(t) \leq v_j(t)$  for  $j^* \leq j \leq i_t$ . Without loss of generality, we may assume that  $u_{i_t-1}(t) < v_{i_t-1}(t)$ . Then  $\dot{u}_{i_t}(t) < \dot{v}_{i_t}(t)$  and hence

$$u_{i_t}(t - \epsilon) > v_{i_t}(t - \epsilon) \quad \text{for } 0 < \epsilon \ll 1.$$

Then  $j_{t-\epsilon} > i_t$  and  $u_{i_t-1}(t - \epsilon) > v_{i_t-1}(t - \epsilon)$  for  $0 < \epsilon \ll 1$ . This implies that  $u_{i_t-1}(t) \geq v_{i_t-1}(t)$ , which is a contradiction. Therefore,  $u_j(t) \geq v_j(t)$  for  $j \leq i_t$ . Similarly, we can prove that  $u_j(t) \leq v_j(t)$  for  $j > i_t$ . The lemma is thus proved.  $\square$

Next, we prove Theorem 1.3.

*Proof of Theorem 1.3.* (1) Let  $c(t)$ ,  $\bar{v}$ ,  $\underline{v}$  be as in Lemma 5.2 with  $\bar{c}_{\inf} = \gamma$ . For  $\tau \geq 0$ , let  $v^\tau(x, t)$ ,  $t \geq -\tau$  be the solution of

$$\begin{cases} \partial_t v - H v = v f(t, v), & x \in \mathbb{R}, t > -\tau, \\ v(x, -\tau) = \bar{v}(x, -\tau), & x \in \mathbb{R} \end{cases} \quad (5.3)$$

and let  $v_\tau(x, t)$ ,  $t \geq -\tau$  be the solution of

$$\begin{cases} \partial_t v - H v = v f(t, v), & x \in \mathbb{R}, t > -\tau, \\ v(x, -\tau) = \underline{v}(x, -\tau), & x \in \mathbb{R}. \end{cases} \quad (5.4)$$

By Lemma 5.2, we have

$$\underline{v}(x, t) \leq v_\tau(x, t) \leq v^\tau(x, t) \leq \bar{v}(x, t) \quad \forall x \in \mathbb{R}, t \geq -\tau.$$

Since  $\bar{v}(x, t)$  is nonincreasing in  $x$ ,  $v^\tau(x, t)$  is nonincreasing in  $x$ . Moreover, if  $\tau_2 > \tau_1 \geq 0$ , then

$$v_{\tau_1}(x, t) \leq v_{\tau_2}(x, t) \leq v^{\tau_2}(x, t) \leq v^{\tau_1}(x, t)$$

for all  $(x, t) \in \mathbb{R} \times [-\tau_1, \infty]$ . Therefore  $\lim_{\tau \rightarrow \infty} v_\tau(x, t)$  and  $\lim_{\tau \rightarrow \infty} v^\tau(x, t)$  exist, and  $\lim_{\tau \rightarrow \infty} v_\tau(x, t)$  is lower-semicontinuous and  $\lim_{\tau \rightarrow \infty} v^\tau(x, t)$  is upper-semicontinuous. Let  $v^\pm : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be such that

$$\lim_{\tau \rightarrow \infty} v_\tau(x, t) = v^-(x, t) \quad \text{pointwise in } (x, t) \in \mathbb{R} \times \mathbb{R}$$

and

$$\lim_{\tau \rightarrow \infty} v^\tau(x, t) = v^+(x, t) \quad \text{pointwise in } (x, t) \in \mathbb{R} \times \mathbb{R}.$$

Then

$$\liminf_{(x, t) \rightarrow (x_0, t_0)} v^-(x, t) \geq v^-(x_0, t_0), \quad \limsup_{(x, t) \rightarrow (x_0, t_0)} v^+(x, t) \leq v^+(x_0, t_0) \quad (5.5)$$

for any  $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}$ . Since we have for any  $x \in \mathbb{R}$  and  $t \geq -\tau$ ,

$$v_\tau(x, t) = v_\tau(x, 0) + \int_0^t H v_\tau(x, s) ds + \int_0^t v_\tau f(s, v_\tau) ds$$

and

$$v^\tau(x, t) = v^\tau(x, 0) + \int_0^t H v^\tau(x, s) ds + \int_0^t v^\tau f(s, v^\tau) ds.$$

Let  $\tau \rightarrow \infty$ , it follows from the dominated convergence theorem that for any  $x \in \mathbb{R}$  and  $t \in \mathbb{R}$ ,

$$v^\pm(x, t) = v^\pm(x, 0) + \int_0^t H v^\pm(x, s) ds + \int_0^t v^\pm f(s, v^\pm) ds.$$

Then we find that  $v^\pm(x, t)$  is differentiable in  $t$  and satisfies

$$\partial_t v^\pm = H v^\pm + v^\pm f(t, v^\pm), \quad x \in \mathbb{R}, t \in \mathbb{R}.$$

By Lemma 5.2, we have

$$\underline{v}(x, t) \leq v^-(x, t) \leq v^+(x, t) \leq \bar{v}(x, t) \quad \forall x \in \mathbb{R}, t \in \mathbb{R}.$$

This implies that  $\rho(v^-(\cdot, t), v^+(\cdot, t))$  is bounded in  $t$ , and if  $v^-(x, t) \neq v^+(x, t)$ , then  $u_1(x, t) = v^-(x, t)$  and  $u_2(x, t) = v^+(x, t)$  satisfy the conditions in Proposition 2.3(3). Assume that  $v^-(x, t) \neq v^+(x, t)$ . Then by Proposition 2.3(3), we have that for any  $\tau > 0$  and  $T \in \mathbb{R}$ , there is  $\delta > 0$  such that

$$\rho(v^-(\cdot, s + \tau), v^+(\cdot, s + \tau)) < \rho(v^-(\cdot, s), v^+(\cdot, s)) - \delta$$

for  $s \leq T$ . This implies that

$$\begin{aligned} \rho(v^-(\cdot, -n), v^+(\cdot, -n)) &= \rho(v^-(\cdot, -(n + \tau) + \tau), v^+(\cdot, -(n + \tau) + \tau)) \\ &< \rho(v^-(\cdot, -(n + \tau)), v^+(\cdot, -(n + \tau))) - \delta \\ &< \rho(v^-(\cdot, -(n + k\tau)), v^+(\cdot, -(n + k\tau))) - k\delta \end{aligned}$$

for all  $n \in \mathbb{N}$  with  $n \geq -T$  and  $k \in \mathbb{N}$ . This is a contradiction since  $\rho(v^-(\cdot, -(n + k\tau)), v^+(\cdot, -(n + k\tau))) - k\delta < 0$  for  $k \gg 1$ . Therefore,

$$v^-(x, t) = v^+(x, t).$$

Then by (5.5),  $v(x, t) := v^+(x, t)$  is continuous in  $x \in \mathbb{R}$  and  $t \in \mathbb{R}$  and is nonincreasing in  $x$ . Moreover, we have

$$\lim_{x \rightarrow \infty} \frac{v(x + \int_0^t c(\tau) d\tau, t)}{e^{-\mu x}} = 1 \quad \text{uniformly in } t \in \mathbb{R}.$$

By Lemma 5.2,

$$\delta_0 := \inf_{x \leq 0, t \in \mathbb{R}} v(x + \int_0^t c(\tau) d\tau, t) > 0.$$



By Theorem 1.1(1) and Proposition 2.2, we have

$$\lim_{x \rightarrow -\infty} (v(x + \int_0^t c(\tau) d\tau, t) - u^+(t)) = 0 \quad \text{uniformly in } t \in \mathbb{R}.$$

Then we can get the desired function  $\phi(x, t) = v(x + \int_0^t c(\tau) d\tau, t)$ .

(2) By Theorem 1.2,  $\tilde{c}_* \geq \tilde{c}_0^-$ . Assume that  $\tilde{c}_* > \tilde{c}_0^-$ . Fix  $\gamma, c',$  and  $c''$  such that

$$\tilde{c}_0^- < \gamma < c' < c'' < \tilde{c}_*.$$

Observe that  $\tilde{c}_0^- > 0$ . By Theorem 1.2, for any  $u^0 \in l_0^\infty(\mathbb{Z})$ ,

$$\limsup_{|i| \leq c''t, t \rightarrow \infty} |u_i(t + s; s, u^0) - u^+(t + s)| = 0 \quad (5.6)$$

uniformly in  $s \in \mathbb{R}$ .

Let  $\phi(x, t) = v(x + \int_0^t c(\tau) d\tau, t)$  be as in (1). Let

$$u_i^s = \phi(i + [\int_0^s c(\tau) d\tau], s) \quad \forall s \in \mathbb{R}.$$

By (1), there is  $u^0 \in l_0^\infty(\mathbb{Z})$  such that

$$u^0 \leq u^s \quad \forall s \in \mathbb{R}.$$

Hence

$$u_i(t; s, u^0) \leq u_i(t; s, u^s) \quad \forall i \in \mathbb{Z}, s \in \mathbb{R}, t \geq s.$$

This together with (5.6) implies that

$$\limsup_{|i| \leq c''t, t \rightarrow \infty} |u_i(t + s; s, u^s) - u^+(t + s)| = 0 \quad (5.7)$$

uniformly in  $s \in \mathbb{R}$ .

By (1) again,

$$u_i(t; s, u^s) = \phi(i - \int_0^t c(\tau) d\tau + [\int_0^s c(\tau) d\tau], t) \leq \phi(i - \int_s^t c(\tau) d\tau - 1, t)$$

and then

$$\limsup_{i \geq (c'' - c')t + \int_s^{t+s} c(\tau) d\tau, t \rightarrow \infty} u_i(t + s; s, u^s) = 0 \quad (5.8)$$

uniformly in  $s \in \mathbb{R}$ . It follows from (5.7) and (5.8) that

$$\bar{c}_{\inf} \geq c' > \gamma,$$

which is a contradiction. Therefore,  $\tilde{c}_* = \tilde{c}_0^-$ . □

**Remark 5.2.** (1) If  $f(t, u) \equiv f(u)$ , then  $\bar{\phi}(x, t) \equiv \bar{\phi}(x)$ . We claim that  $\phi(x, t) \equiv \phi(x)$ . In fact, when  $f(t, u) \equiv f(u)$ , we have

$$\int_{t_1}^{t_2} c(s) ds = \int_0^{t_2 - t_1} c(s) ds \quad \forall t_1, t_2 \in \mathbb{R}$$

and

$$u(x, t; s, u_0) = u(x, 0; s - t, u_0) \quad \forall t \geq s, u_0 \in l^{\infty, +}(\mathbb{R}).$$

We then have

$$\begin{aligned}
\phi(x, t) &= v^+(x + \int_0^t c(\tau) d\tau, t) \\
&= \lim_{\tau \rightarrow \infty} v^\tau(x + \int_0^t c(s) ds, t) \\
&= \lim_{\tau \rightarrow \infty} u(x + \int_0^t c(s) ds, t; -\tau, \bar{v}(\cdot, -\tau)) \\
&= \lim_{\tau \rightarrow \infty} u(x, t; -\tau, \bar{\phi}(\cdot + \int_0^t c(s) ds - \int_0^{-\tau} c(s) ds)) \\
&= \lim_{\tau \rightarrow \infty} u(x, t; -\tau, \bar{\phi}(\cdot - \int_0^{-(t+\tau)} c(s) ds)) \\
&= \lim_{\tau \rightarrow \infty} u(x, 0; -(t+\tau), \bar{\phi}(\cdot - \int_0^{-(t+\tau)} c(s) ds)) \\
&= v^+(x, 0) = \phi(x) \quad \forall t \in \mathbb{R}.
\end{aligned}$$

The claim thus follows.

(2) If  $f(t, u) = f(t+T, u)$ , then  $\bar{\phi}(x, t+T) = \bar{\phi}(x, t)$ . We claim that  $\phi(x, t+T) = \phi(x, t)$ . In fact, when  $f(t+T, u) = f(t, u)$ , we have

$$\int_T^{t+T} c(s) ds = \int_0^t c(s) ds \quad \forall t \in \mathbb{R}$$

and

$$u(x, t+mT; nT, u_0) = u(x, t; (n-m)T, u_0) \quad \forall n, m \in \mathbb{Z}.$$

We then have

$$\begin{aligned}
\phi(x, t+T) &= v^+(x + \int_0^{t+T} c(s) ds, t+T) \\
&= \lim_{n \rightarrow \infty} v^{nT}(x + \int_0^{t+T} c(s) ds, t+T) \\
&= \lim_{n \rightarrow \infty} u(x + \int_0^{t+T} c(s) ds, t+T; -nT, \bar{v}(\cdot, -nT)) \\
&= \lim_{n \rightarrow \infty} u(x + \int_0^{t+T} c(s) ds, t+T; -nT, \bar{\phi}(\cdot - \int_0^{-nT} c(s) ds, -nT)) \\
&= \lim_{n \rightarrow \infty} u(x + \int_0^t c(s) ds, t+T; -nT, \bar{\phi}(\cdot + \int_0^T c(s) ds - \int_0^{-nT} c(s) ds, -nT)) \\
&= \lim_{n \rightarrow \infty} u(x + \int_0^t c(s) ds, t; -(n+1)T, \bar{\phi}(\cdot - \int_0^{-(n+1)T} c(s) ds, -(n+1)T)) \\
&= v^+(x + \int_0^t c(s) ds, t) = \phi(x, t) \quad \forall t \in \mathbb{R}.
\end{aligned}$$

The claim thus also holds.

We now prove Theorem 1.4.

*Proof of Theorem 1.4.* First of all, let  $\mu^*$  be as in Lemma 5.1. For given  $0 < \mu \leq \mu^*$ , let  $c_\mu(t) = \frac{e^{-\mu} + e^\mu - 2 + f(t, 0)}{\mu}$ ,  $\varphi_\mu(x) = e^{-\mu x}$ , and  $\bar{\phi}_\mu(x, t) = \min\{\varphi_\mu(x), u^+(t)\}$ . Let

$$\bar{v}_\mu(x, t) = \bar{\phi}_\mu(x - \int_0^t c_\mu(s) ds, t).$$

By Theorem 1.3, for given  $0 < \mu < \mu^*$ , for each  $t \in \mathbb{R}$ ,

$$v_\mu(x, t) := \lim_{\tau \rightarrow \infty} u(x, t; -\tau, \bar{v}_\mu(\cdot, -\tau))$$

uniformly in  $x \in \mathbb{R}$ , and

$$\lim_{x \rightarrow -\infty} v_\mu(x + \int_0^t c_\mu(s) ds, t) = u^+(t), \quad \lim_{x \rightarrow \infty} v_\mu(x + \int_0^t c_\mu(s) ds, t) = 0 \quad (5.9)$$

uniformly in  $t \in \mathbb{R}$ .

Next, for given  $0 < \mu < \mu^*$ ,  $n \in \mathbb{N}$ , and  $t > -n$ , let  $x(\mu, t, n)$  be such that

$$u(x(\mu, t, n), t; -n, \bar{\phi}_\mu(\cdot, -n)) = u(0, t; -n, \bar{\phi}_\mu(\cdot + x(\mu, t, n), -n)) = \frac{u^+(t)}{2}.$$

Note that

$$v_\mu(x + \int_0^t c_\mu(s) ds, t) = \lim_{n \rightarrow \infty} u(x, t; -n, \bar{\phi}_\mu(\cdot + \int_{-n}^t c_\mu(s) ds, -n)).$$

By (5.9), we have that there is  $M > 0$  such that for any  $t \in \mathbb{R}$ ,

$$|x(\mu, t, n) - \int_{-n}^t c_\mu(s) ds| \leq M \quad \forall n \gg 1. \quad (5.10)$$

Moreover, for any  $\epsilon > 0$ , there is  $\tilde{M}_\epsilon > 0$  such that for any  $t \in \mathbb{R}$ ,

$$J^+(t, \mu, n) - J^-(t, \mu, n) < \tilde{M}_\epsilon \quad \forall n \gg 1, \quad (5.11)$$

where  $J^\pm(t, \mu, n)$  are such that

$$u(j + x(\mu, t, n), t; -n, \bar{\phi}_\mu(\cdot, -n)) \begin{cases} \geq u^+(t) - \epsilon & \text{for } j \leq J^-(t, \mu, n) \\ \leq \epsilon & \text{for } j \geq J^+(t, \mu, n). \end{cases}$$

Now, for given  $n \in \mathbb{N}$  and  $0 < \mu < \mu^*$ , there are  $x(\mu^*, n)$  and  $x(\mu, n) := x(\mu, 0, n)$  such that

$$u(x(\mu^*, n), 0; -n, \bar{\phi}_{\mu^*}(\cdot, -n)) = \frac{u^+(0)}{2}, \quad u(x(\mu, n), 0; -n, \bar{\phi}_\mu(\cdot, -n)) = \frac{u^+(0)}{2}. \quad (5.12)$$

Note that

$$u(x + x(\mu^*, n), t; -n, \bar{\phi}_{\mu^*}(\cdot, -n)) = u(x, t; -n, \bar{\phi}_{\mu^*}(\cdot + x(\mu^*, n), -n))$$

and

$$u(x + x(\mu, n), t; -n, \bar{\phi}_\mu(\cdot, -n)) = u(x, t; -n, \bar{\phi}_\mu(\cdot + x(\mu, n), -n))$$

for  $t \geq -n$  and  $x \in \mathbb{R}$ . Note also that there is  $j_n \in \mathbb{Z}$  such that

$$\bar{\phi}_{\mu^*}(j + x(\mu^*, n), -n) \begin{cases} \geq \bar{\phi}_{\mu}(j + x(\mu, n), -n) & \text{for } j \leq j_n \\ < \bar{\phi}_{\mu}(j + x(\mu, n), -n) & \text{for } j > j_n. \end{cases} \quad (5.13)$$

By (5.12), (5.13), and Lemma 5.3,

$$u(j + x(\mu^*, n), 0; -n, \bar{\phi}_{\mu^*}(\cdot, -n)) \begin{cases} \geq u(j + x(\mu, n), 0; -n, \bar{\phi}_{\mu}(\cdot, -n)) & \text{for } j \leq 0 \\ \leq u(j + x(\mu, n), 0; -n, \bar{\phi}_{\mu}(\cdot, -n)) & \text{for } j > 0. \end{cases} \quad (5.14)$$

Note that there is  $n_k \rightarrow \infty$  such that  $\lim_{n_k \rightarrow \infty} u(j + x(\mu^*, n_k), 0; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k))$  exists for all  $j \in \mathbb{Z}$ . Without loss of generality, we may assume that  $\lim_{n_k \rightarrow \infty} u(j + x(\mu^*, n_k), -m; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k))$  exists for all  $j \in \mathbb{Z}$  and  $m \in \mathbb{N}$ . Let

$$u_j^{-m,*} = \lim_{n_k \rightarrow \infty} u(j + x(\mu^*, n_k), -m; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) \quad \forall j \in \mathbb{Z}.$$

By Proposition 2.2,

$$u(j, 0; -m, u_j^{-m,*}) = \lim_{n_k \rightarrow \infty} u(j + x(\mu^*, n_k), 0; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) = u_j^{0,*} \quad \forall j \in \mathbb{Z}, \quad m \in \mathbb{N},$$

and then

$$u(j, t; 0, u_j^{0,*}) = \lim_{n_k \rightarrow \infty} u(j + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) \quad \forall j \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

Hence  $u(j, t; 0, u_j^{0,*})$  is an entire solution of (1.1). It is clear that  $u(j, t; 0, u_j^{0,*})$  is nonincreasing in  $j \in \mathbb{Z}$ .

We claim that  $u(j, t; 0, u_j^{0,*})$  is a transition front solution satisfying the properties in Theorem 1.4.

To prove the claim, we first prove that

$$\lim_{j \rightarrow -\infty} u(j, t; 0, u_j^{0,*}) = u^+(t), \quad \lim_{j \rightarrow \infty} u(j, t; 0, u_j^{0,*}) = 0 \quad \forall t \in \mathbb{R}. \quad (5.15)$$

Note that without loss of generality, we may assume  $\lim_{n_k \rightarrow \infty} u(j, 0; -n_k, \bar{\phi}_{\mu}(\cdot + x(\mu, n_k), -n_k))$  exists for all  $j \in \mathbb{Z}$ . Let

$$\tilde{v}_j^{\mu} = \lim_{n_k \rightarrow \infty} u(j, 0; -n_k, \bar{\phi}_{\mu}(\cdot + x(\mu, n_k), -n_k)) \quad \forall j \in \mathbb{Z}.$$

By Proposition 2.2 again,

$$\tilde{v}_{\mu}(j, t) := u(j, t; 0, \tilde{v}^{\mu}) = \lim_{n_k \rightarrow \infty} u(j, t; -n_k, \bar{\phi}_{\mu}(\cdot + x(\mu, n_k), -n_k)) \quad \forall j \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

Note that

$$v_{\mu}(x, t) = \lim_{n \rightarrow \infty} u(x, t; -n, \bar{\phi}_{\mu}(\cdot - \int_0^{-n} c_{\mu}(\tau) d\tau, -n)) \quad \forall x \in \mathbb{R}.$$

By (5.12),

$$\tilde{v}_{\mu}(0, 0) = \frac{u^+(0)}{2}.$$

By (5.10),  $x(\mu, n_k) + \int_0^{-n_k} c_{\mu}(\tau) d\tau$  is bounded. Hence

$$\lim_{j \rightarrow -\infty} \tilde{v}_{\mu}(j, 0) = u^+(0), \quad \lim_{j \rightarrow \infty} \tilde{v}_{\mu}(j, 0) = 0.$$

It then follows from the monotonicity of  $\tilde{v}_\mu(j, t)$  in  $j \in \mathbb{Z}$  that

$$\lim_{j \rightarrow -\infty} \tilde{v}_\mu(j, t) = u^+(t), \quad \lim_{j \rightarrow \infty} \tilde{v}_\mu(j, t) = 0 \quad \forall t \in \mathbb{R}.$$

This together with (5.14) implies that (5.15).

Next, let  $J(t) \in \mathbb{Z}$  be such that

$$u(j, t; 0, u^{0,*}) \begin{cases} \geq \frac{u^+(t)}{2} & \text{for } j \leq J(t) \\ < \frac{u^+(t)}{2} & \text{for } j > J(t). \end{cases}$$

By (5.15),  $J(t)$  is well defined for each  $t \in \mathbb{R}$ . We prove that

$$\lim_{j \rightarrow -\infty} u(j + J(t), t; 0, u^{0,*}) = u^+(t), \quad \lim_{j \rightarrow \infty} u(j + J(t), t; 0, u^{0,*}) = 0 \quad (5.16)$$

uniformly in  $t \in \mathbb{R}$ . To end this, fix  $t \in \mathbb{R}$ . Note that there is  $x_{t,n_k}^* \in \mathbb{R}$  such that

$$u(x_{t,n_k}^* + J(t) + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) = \frac{u^+(t)}{2}$$

Then

$$u(1 + J(t) + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) < u(x_{t,n_k}^* + J(t) + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k))$$

for  $k \gg 1$ . Recall that

$$u(x(\mu, t, n_k), t; -n_k, \bar{\phi}_\mu(\cdot, -n_k)) = \frac{u^+(t)}{2}.$$

By the similar arguments for (5.14),

$$\begin{aligned} & u(j + x_{t,n_k}^* + J(t) + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) \\ & \begin{cases} \geq u(j + x(\mu, t, n_k), t; -n_k, \bar{\phi}_\mu(\cdot, -n_k)) & \forall j \leq 0 \\ \leq u(j + x(\mu, t, n_k), t; -n_k, \bar{\phi}_\mu(\cdot, -n_k)) & \forall j > 0. \end{cases} \end{aligned} \quad (5.17)$$

This together with (5.11) implies that there is  $J^*$  such that

$$u(J^* + x_{t,n_k}^* + J(t) + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) < \frac{u^+(t)}{4} \quad \forall k \gg 1.$$

It then follows that

$$u(J(t) + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) > u(J^* + x_{t,n_k}^* + J(t) + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k))$$

for  $k \gg 1$ . Therefore

$$x_{t,n_k}^* + J(t) - 1 + x(\mu^*, n_k) < J(t) + x(\mu^*, n_k) < J^* + x_{t,n_k}^* + J(t) + x(\mu^*, n_k)$$

and then

$$\begin{aligned} & u(j + J^* + x_{t,n_k}^* + J(t) + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) \\ & \leq u(j + J(t) + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) \\ & \leq u(j + x_{t,n_k}^* + J(t) - 1 + x(\mu^*, n_k), t; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) \end{aligned} \quad (5.18)$$

for  $k \gg 1$ . This together with (5.11) and (5.17) implies (5.16).

We now prove that

$$\liminf_{t-s \rightarrow \infty} \frac{J(t) - J(s)}{t - s} = \tilde{c}_0^-. \quad (5.19)$$

By Theorem 1.2, we have

$$\liminf_{t-s \rightarrow \infty} \frac{J(t) - J(s)}{t - s} \geq \tilde{c}_0^-.$$

Fix  $s < t$  and  $0 < \mu < \mu^*$ . By (5.17) and (5.18),

$$u(j + J(s) + x(\mu^*, n_k), s; -n_k, \bar{\phi}_{\mu^*}(\cdot, -n_k)) \leq u(j - 1 + x(\mu, s, n_k), s; -n_k, \bar{\phi}_{\mu}(\cdot, -n_k)) \quad (5.20)$$

for  $j \geq 2$  and  $k \gg 1$ . By (5.10), without loss of generality, we may assume that there is  $j_s$ , which is bounded in  $s$  such that

$$\lim_{n_k \rightarrow \infty} u(j - 1 + x(\mu, s, n_k), s; -n_k, \bar{\phi}_{\mu}(\cdot, -n_k)) = v_{\mu}(j + j_s + \int_0^s c_{\mu}(r)dr, s) \quad \forall j \in \mathbb{Z}.$$

This together with (5.20) implies that

$$u(j + J(s), s; 0, u^{0,*}) \leq v_{\mu}(j + j_s + \int_0^s c_{\mu}(r)dr, s) \quad \forall j \geq 2. \quad (5.21)$$

Note that

$$\inf_{s \in \mathbb{R}} v_{\mu}(1 + j_s + \int_0^s c_{\mu}(r)dr, s) > 0.$$

Let

$$\kappa = \begin{cases} \inf_{j \leq 0} \frac{v_{\mu}(j_s + \int_0^s c_{\mu}(r)dr, s)}{u(j + J(s), s; 0, u^{0,*})} & \text{if } u(1 + J(s), s; 0, u^{0,*}) \leq v_{\mu}(1 + j_s + \int_0^s c_{\mu}(r)dr, s) \\ \inf_{j \leq 1} \frac{v_{\mu}(1 + j_s + \int_0^s c_{\mu}(r)dr, s)}{u(j + J(s), s; 0, u^{0,*})} & \text{if } u(1 + J(s), s; 0, u^{0,*}) > v_{\mu}(1 + j_s + \int_0^s c_{\mu}(r)dr, s). \end{cases}$$

Then  $0 < \kappa < 1$  and

$$\kappa u(j + J(s), s; 0, u^{0,*}) \leq v_{\mu}(j + j_s + \int_0^s c_{\mu}(r)dr, s) \quad \forall j \in \mathbb{Z}.$$

Observe that for  $t \geq s$ ,

$$\kappa u(j + J(s), t; 0, u^{0,*}) \leq u(j, t; s, \kappa u(\cdot + J(s), s; 0, u^{0,*})) \leq v_{\mu}(j + j_s + \int_0^s c_{\mu}(r)dr, t) \quad \forall j \in \mathbb{Z}.$$

This implies that there is  $L > 0$  independent of  $s$  and  $t$  such that

$$J(t) - J(s) \leq \int_s^t c_{\mu}(r)dr + L.$$

It then follows that

$$\liminf_{t-s \rightarrow \infty} \frac{J(t) - J(s)}{t - s} \leq \liminf_{t-s \rightarrow \infty} \frac{\int_s^t c_{\mu}(r)dr}{t - s}$$

for any  $0 < \mu < \mu^*$ . Then by Theorem 1.3,

$$\liminf_{t-s \rightarrow \infty} \frac{J(t) - J(s)}{t - s} \leq \tilde{c}_0^-.$$

Therefore, (5.19) holds and Theorem 1.4 is thus proved.  $\square$

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